7 "New way" integrals

k - field ; $\mathbb{A}$ - its adele ring, P - set of places of k.

G - reductive group/k

(π, V) irr. cuspidal aut. rep^n of G, ρ f. dim. rep^n of ${}^LG(\mathbb{C})$ $\qquad \pi \cong \otimes \pi_v$

We want to give an integral representation to $L^{S'}(s, \pi, \rho)$

Rankin - Selberg method

⓪ Cook up a global integral (+ Unfolding) $Z(s, \varphi, \ldots)$

① Factorizability $Z(s, \otimes \varphi_v, \ldots) = \prod_{v \in P} Z_v(s, \varphi_v, \ldots)$

② Unramified calculation $\exists$ unr. data : $Z_v(s, \_) = L(s, \pi, \rho)$

Example: Let (π, V) be an irr. cusp. aut. rep^n of $GL_2(\mathbb{A})$.

For $\varphi \in V_\pi$ consider: $Z(s, \varphi) = \int_{k^\times \backslash \mathbb{A}^\times} \varphi\begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s - \frac{1}{2}} d^\times a$

Unfolding:

By Fourier analysis $\varphi(g) = \sum_{\psi \text{ characters of } k \backslash \mathbb{A}} \int_{k \backslash \mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(x)\, dx =$

Fix nondegenerate char ψ of $k\backslash\mathbb{A}$; any other char is $\psi_\gamma(x) = \psi(\gamma x)$

$= \sum_{\gamma \in k^\times} \int_{k \backslash \mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(\gamma x)\, dx = \sum_{\gamma \in k^\times} \int_{k \backslash \mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \psi(x)\, dx$

($\uparrow$ cuspidality)

denoting $W_\varphi^\psi(g) = \int_{k \backslash \mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(x)\, dx$ we get

$$Z(s, \varphi) = \int_{k^\times \backslash \mathbb{A}^\times} \sum_{\gamma \in k^\times} W_\varphi^\psi\begin{pmatrix} \gamma a & \\ & 1 \end{pmatrix} |a|^{s - \frac{1}{2}} d^\times a = \int_{\mathbb{A}^\times} W_\varphi^\psi\begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s - \frac{1}{2}} d^\times a$$

Remark: $W_\varphi^\psi(g) = W^\psi(g \cdot \varphi)$, $W^\psi \in \mathrm{Hom}_N(\pi, \mathbb{C}_\psi)$

Def: A model $l \in \pi^\vee$ is called factorizable if $l = \otimes_{v \in P} l_v$

this is true for $\mathrm{Hom}_N(\pi, \mathbb{C}_\psi)$ but we don't want to use it.

We will use (and show):

Prop: Let π_v, ψ_v be unr. then for any $l \in \mathrm{Hom}_{N(k_v)}(\pi_v, \mathbb{C}_{\psi_v})$

$$Z_v(s, \varphi_v) = \int_{k_v^\times} l\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} \cdot \varphi_v\right) |a|^{s - \frac{1}{2}} d^\times a = l(\varphi_v) L(s, \pi_v, St)$$

Thm: $\exists$ data φ and a finite set $S \subseteq P$ s.t.

$$Z(s, \varphi) = G_S(s, \varphi) L^S(s, \pi, St)$$

Proof: $Z(s,\varphi) = \lim_{\substack{\Omega\to\mathcal{P}\\ |\Omega|<\infty \\ S\subseteq\Omega}} \int_{\mathbb{A}_\Omega^\times} W_\varphi^\psi\begin{pmatrix}a&\\&1\end{pmatrix}|a|^{s-\frac12}\,d^\times a$ $\qquad \mathbb{A}_\Omega=\prod_{v\in\Omega}k_v$, $\hat{\mathbb{A}}$

for $v\notin\Omega$:

$$\int_{\mathbb{A}_{\Omega\cup\{v\}}} W_\varphi^\psi\begin{pmatrix}a&\\&1\end{pmatrix}|a|^{s-\frac12}d^\times a = \int_{\mathbb{A}_\Omega^\times}\int_{k_v^\times} W_1^\psi\left(\begin{pmatrix}a_v&\\&1\end{pmatrix}\begin{pmatrix}a_\Omega&\\&1\end{pmatrix}\cdot\varphi\right)|a_v|^{s-\frac12}d^\times a_v\,|a_\Omega|^{s-\frac12}d^\times a_\Omega$$

$$= L_v(s,\pi_v,st)\int_{\mathbb{A}_\Omega^\times} W_\varphi^\psi\begin{pmatrix}a&\\&1\end{pmatrix}|a|^{s-\frac12}d^\times a$$

induction $\Rightarrow Z(s,\varphi) = \lim_{\Omega} \prod_{v\in\Omega\setminus S'} L_v(s,\pi_v,st)\underbrace{\int_{\mathbb{A}_{S'}^\times} W_\varphi^\psi\begin{pmatrix}a&\\&1\end{pmatrix}|a|^{s-\frac12}d^\times a}_{G_{S'}(s,\varphi)} =$

$$= G_{S'}(s,\varphi)\,L^{S'}(s,\pi,st)$$

In order to prove the local statement, recall Satake isomorphism

$G=G(k_v)$, $K=G(\mathcal{O}_v)$

$$\begin{tikzcd} \mathcal{H}(G,K) \arrow[rr, "\simeq", "\{A_\lambda\}\longleftrightarrow\{V_\lambda\}"'] \arrow[dr, "S_{Hecke,\pi}"'] & & Rep({}^LG(\mathbb{C})) \arrow[dl, "S_{Rep,\pi}"] \\ & \mathbb{C} & \end{tikzcd}$$

SPECTRAL DECOMPOS.

$\forall\,\pi$ unr. rep^n of G $\qquad S_{Hecke,\pi}$: $\mathcal{H}\circlearrowright V^K$ $\qquad S_{Rep,\pi}(\rho,V)=tr_V\,\rho(t_\pi)$

Lemma: $\exists!\ \Delta(s,\cdot)\in\mathcal{H}[[q^{-s}]]$ s.t. $\forall\, l\in\pi^\vee$: $\int_G \Delta(s,g)\,l(g\cdot v^\circ)\,dg = l(v^\circ)L(s,\pi,st)$

(GL_2) **Proof:** assume $t_\pi = \begin{pmatrix}x_1&\\&x_2\end{pmatrix}$

$$L(s,\pi,st) = (1-x_1q^{-s})^{-1}(1-x_2q^{-s})^{-1} = \sum_{k=0}^\infty (x_1q^{-s})^k\sum_{k=0}^\infty(x_2q^{-s})^k =$$

$$= \sum_{n=0}^\infty\Big(\sum_{i_1+i_2=n}x_1^{i_1}x_2^{i_2}\Big)q^{-ns} = \sum_{n=0}^\infty tr\,Sym^n(t_\pi)\,q^{-ns} =$$

$\boxed{\int f(g)\,l(g\cdot v^\circ)\,dg = l(v^\circ)\,S_{Hecke,\pi}(f)}$

$$= \sum_{n=0}^\infty S_{Rep,\pi}(V_{n\omega_1})\,q^{-ns} = S_{Hecke,\pi}\Big(\underbrace{\sum_{n=0}^\infty A_{n\omega_1}q^{-ns}}_{\Delta(s,\cdot)}\Big)$$

$\boxed{\Delta(s,\cdot)\text{ is called generating f. of } L(s,\cdot,st)}$

Rem: $\omega_1\begin{pmatrix}a&\\&b\end{pmatrix}=a$, $\omega_2\begin{pmatrix}a&\\&b\end{pmatrix}=ab$ (fund. weights.)

Proof of local statement: For any $l\in Hom_N(\pi,\mathbb{C}_\psi)$

$$\int_G \Delta(s,g)\,l(g\cdot v^\circ)\,dg = \int_{N\backslash G}\int_N \Delta(s,ng)\underbrace{l(ng\cdot v^\circ)}_{\psi(n)\,l(g\cdot v^\circ)}\,dn\,dg = \int_{N\backslash G}\Delta^\psi(s,g)\,l(g\cdot v^\circ)\,dg =$$

$$= \int_T \Delta^\psi(s,t)\,l(t\cdot v^\circ)\,dt = \qquad \text{Casselman–Shalika: } A_\lambda^{\psi,N} = \delta_B^{-1/2}\,\mathbb{1}_{(N,\psi)\lambda(\varpi)K}$$

$$= \sum_{n=0}^\infty q^{-ns}\int \mathbb{1}_{(N,\psi)\begin{pmatrix}\varpi^n&\\&1\end{pmatrix}K}(t)\,\delta_B^{-1/2}(t)\,l(t\cdot v^\circ)\,dt = \sum_{k=0}^\infty q^{-ks}q^{\frac k2}\int_{|a|=q^{-k}} l\left(\begin{pmatrix}a&\\&1\end{pmatrix}\cdot v^\circ\right)da =$$

$$= \int_{k_v^\times} l\left(\begin{pmatrix}a&\\&1\end{pmatrix}\cdot v^\circ\right)\cdot|a|^{s-\frac12}\,d^\times a \qquad \text{because } l\left(\begin{pmatrix}a&\\&1\end{pmatrix}v^\circ\right)=0 \text{ if } a\notin\mathcal{O}$$

Two others examples:

[Bump, Furasawa, Ginzburg] $G = GL_{rn}$

$$\begin{pmatrix} r\times r & r\times r & r & \\ & & & \\ & & & \\ \end{pmatrix}$$

$$\mathcal{U} = \begin{pmatrix} I & * & * & * & * \\ & I & * & * & * \\ & & I & * & * \\ & & & \ddots & * \\ & & & & I \end{pmatrix}, \quad \mathcal{U}' = \begin{pmatrix} I & 0 & * & * & * \\ & I & * & * & * \\ & & I & * & * \\ & & & I & * \\ & & & & I \end{pmatrix}$$

Ψ_U - non-degenerate char. of U

$$\int_{GL(r,k)\backslash GL(r,\mathbb{A})} \int_{U'(k)\backslash U'(\mathbb{A})} \varphi\left(u\begin{pmatrix} g & \\ & I_{(n-1)r}\end{pmatrix}\right) \overline{\Psi_U(u)}\, |\det g|^{s+r-\frac{rn+1}{2}}\, du\, dg$$

$r=1, n=1 \to$ Tate

$r=1, n=2 \to$ J-L

represent $L^S(s,\pi,st)$ $\quad \varphi \in \Pi$

$r=1,\ n\geq 1 \quad$ J-PS-Shalika

The integral unfolds to this model

$$H(\varphi) = \int_{U(k)\backslash U(\mathbb{A})} \varphi(u)\, \overline{\Psi_U(u)}\, du$$

$$\Delta(s,g) = \begin{cases} |\det g|, & g \in M_{rn}(\mathcal{O}_v) \\ 0, & \text{otherwise} \end{cases}$$

[PS, Rallis] $G = Sp_{2n}$ $\quad \varphi \in \Pi$

$$\int \varphi(g)\, \Theta_T(g)\, E(g,s)\, dg$$

E - degenerate Eisenstein series of Sp_{2n}

$U = \begin{pmatrix} I & Y \\ & I \end{pmatrix}$

T - quad. form $\rightsquigarrow$ quad. character on k^* $\chi_T \rightsquigarrow$ suitable par. $t_T = \pm 1$

$L(s, \pi \otimes \chi_T, \underset{Sp_{2n+1}}{\underset{\uparrow}{st}} \otimes \underset{GL_1}{\underset{\uparrow}{st}})$ $\quad \Theta$ - associated to $Sp_{2n} \otimes \mathcal{O}(T) \hookrightarrow \widetilde{Sp}_{2n^2}$

Example: $G = G_2$ $\alpha \Rrightarrow \beta$

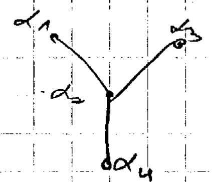
$\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$

$P = MU$, $M = GL_2(\alpha)$

$\{\text{non-deg. char. on } \Psi\} \longleftrightarrow \{\text{cubic algebras } /k\}$

we pick Ψ to be the one corresponding to $k \times k \times k$

$H = Spin_8$



$\circlearrowleft S^3$

$H^{S_3} = G_2$

$P_H = M_H U_H$ $M_H = GL_2(\alpha_1) \times GL_2(\alpha_3) \times GL_2(\alpha_4)$

~~Let~~ $E(g,s,f)$ - Eisenstein series associated to $Ind_{P_H}^H \delta_{P_H}^s$

π - irr. cuspidal aut. rep^n of $G_2(\mathbb{A})$, $\varphi \in \pi$

Consider $Z(s,\varphi,f) = \int_{G_2(k)\backslash G_2(\mathbb{A})} \varphi(g) E(g,s,f)\, dg$

$\xrightarrow{\text{unfolding}} \int_{U(\mathbb{A})\backslash G_2(\mathbb{A})} L^\Psi \varphi(g)\, F(g,s)\, dg$ can be calculated from f, factorizable if f is.

$L^\Psi(g\cdot\varphi) = L^\Psi_\varphi(g) = \int_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\Psi(u)\,du$. $L^\Psi \in Hom_{U(\mathbb{A})}(\pi, \mathbb{C}_\Psi)$ usually inf. dim.

Conjecture: $\exists$ data φ, f, S, s_0 s.t. for $Re\, s > s_0$

$$Z(s,\varphi,f) = L^S(5s-2, \pi, st)\, G_S(s, f_S, \varphi_S) \quad \text{(mero. cont.)}$$

By inductive process, the conj follows from

Conj: For $v \notin S$ and any $l \in Hom_{U(k_v)}(\pi_v, \mathbb{C}_{\Psi_v})$

$$\int_{U(k_v)\backslash G_2(k_v)} l(g\cdot v^0) F_v(g,s)\, dg = l(v^0)\, L(\pi_v, 5s-2, st)$$

$\forall\, l \in Hom_U(\pi, \mathbb{C}_\Psi)$, $l(v^0) = 1$

We do know $\exists\, \Delta(\cdot, s)$ s.t. $\int_{G_2(k_v)} l(g\cdot v^0)\Delta(g,s)\,dg = L(5s-2, \pi_v, st)$ H?

$$\int_{U\backslash G} l(g\cdot v^0)\Delta^{\Psi,U}(g,s)\,dg \quad = \quad \int F_v(g,s)\, l(g\cdot v^0)\, dg$$

Enough to show $\Delta^{\Psi,U} = F_v$

How to do this?

<u>Def</u>: $\mathcal{D}(\alpha^\vee(t_1)\beta^\vee(t_2), s) = |t_1|^{5s+1}$

$$\int \mathcal{D}(g,s)\,\ell(g\cdot v^e)\,dg = \int \Delta(g,s)\, S_\pi(g)\,dg = \left(P_1(q^{-s}) - P_2(q^{-s})\,tr_{st}(t_{\pi_v})\right)\ell(5s-2,\pi_v$$

$$P_s = P_1(q^{-s})\, \overset{1}{A_{trivial}} - P_2(q^{-s}) A_{st} \qquad P_s \in \mathcal{H}$$

$$\mathcal{D}(\cdot, s) = \Delta(\cdot, s) * P_s$$

$$\Rightarrow \mathcal{D}_s^{\psi,\upsilon} = \Delta_s^{\psi,\upsilon} * P_s \overset{?}{=} F_s * P_s$$

<u>Lemma</u>: $*P_s$ is investible.

<u>Proof</u>: $\mathcal{H}$ can be complited to a C^*-alg.

P_s is inv. $\iff$ $1 - \frac{P_2}{P_1} A_{st}$ is

enough to show $\left\| \frac{P_2}{P_1} A_{st} \right\| < 1$

$$\underbrace{\left|\frac{P_2}{P_1}\right|}_{\text{tends to zero as Res grows}} \cdot \|A_{st}\|$$

It remaines to show $\mathcal{D}_s^{\psi,\upsilon} = F_s * P_s$