L-functions & their generating functions

Let • k be a finite extension of $\mathbb{Q}_p$

• $\mathcal{O}$ the ring of integers in k, ϖ a unif. $q = |\mathcal{O}/(\varpi)|$

We start by discussing $G = GL_n(k)$ (p-adic topology)

$K = G(\mathcal{O})$

A rep^n (π, V) of G is called <u>smooth</u> if $\forall v \in V$ $stab_G(v)$ is open in G.

A smooth irr. rep^n (π, V) of G is called unramified if $\dim V^K = 1$

(actually $\dim V^K \leq 1$ for smooth irr. rep^n for $GL_n(\mathcal{O})$)

<u>Local Langlands' conjectures</u> (Harris-Taylor, Henniart 2001)

There is a correspondence

would like to put $Gal(\bar{k}/k)$

$\{$Smooth irr. rep. of $GL_n(k)\} \longleftrightarrow \{$gp. hom. $W_k \times SL_2(\mathbb{C}) \to GL_n(\mathbb{C})\}$

this are rep^n of $W_k \times SL_2(\mathbb{C})$

$$1 \to I \to Gal(\bar{k}/k) \to \hat{\mathbb{Z}} \to 1$$
$$1 \to I \to W_k \to \mathbb{Z} \to 1$$

<u>S.t.</u> * The corr. will behave nicely with: contragredient, tensor products, variation of k, etc. (not $\oplus$)

* cuspidal rep^n $\longleftrightarrow$ irr. rep^n

* The corr. should preserve L-functions.

<u>How to define L-functions?</u>

<u>Galois Side</u>: Given τ and ρ as follows: $W_k \xrightarrow{\tau} GL_n(\mathbb{C}) \xrightarrow{\rho} GL(V)$, $\rho \circ \tau$

$$\begin{array}{ccccc} I & \to & W_k & \to & \mathbb{Z} \\ & & \downarrow{\scriptstyle \rho\circ\tau} & & \cup \\ & & GL(V) & & Fr \end{array}$$

can be lifted in many ways

ρ is an extra information

The action of Fr on V^I is independent from the choice of lifting

<u>Def</u>: $L(s, \tau, \rho) = \det(1 - \rho\circ\tau(Fr)|_{V^I}\, q^{-s})^{-1}$

<u>Remark</u>: V is called unramified if $V = V^I$

p-adic side: We have a correspondence

$$\left\{\begin{matrix}\text{smooth irr. unr.}\\ \text{rep}^n \text{ of } GL_n(k)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Semisimple conj.}\\ \text{classes of } GL_n(\mathbb{C})\end{matrix}\right\}$$

π $\qquad\qquad\qquad$ t_π – called Satake parameter

Def: For a smooth irr. unr. rep^n (π, V) of $GL_n(k)$

let $L(s, \pi, \rho) = \det(\mathbb{1} - \rho(t_\pi) q^{-s})^{-1}$

($\rho: GL_n(\mathbb{C}) \to GL(V)$ f. dim. rep^n)

In the case of unr. rep. LLC are clear. $t_\pi = \Sigma(Fr)$

There is no canonical way to define $L(s,\pi,\rho)$ in general

One way of doing that is via integral representations of $L(s,\pi,\rho)$

Theorem (Godement-Jacquet, 72'): Let (π, V) be a smooth irr. rep^n of G, set

$$Z(s, \Phi, v, v^\vee) = \int_G \Phi(g) |\det g|^s \langle \pi(g) v, v^\vee \rangle dg \qquad \Phi \in \mathcal{S}(M_n(k)), \; v \in V, \; v^\vee \in V^\vee,$$

T. (1) $\exists s_0$ s.t. $Z(s, \Phi, v, v^\vee)$ converge absolutely in $\mathrm{Re}(s) > s_0$

(2) $Z(s, \Phi, v, v^\vee)$ is a rational function of q^{-s}. Moreover

$\{Z(s,\Phi,v,v^\vee) \mid \Phi \in \mathcal{S}(M_n(k)),\, v \in V,\, v^\vee \in V^\vee\}$ admits

a common denominator we call $L(s, \pi, st)$ $\qquad$ $st: GL_n(\mathbb{C}) \hookrightarrow GL$

Indeed, for π unr. this definition coincide with the prev. one.

Where do we get integral representations from? (from global theory)

Spherical Hecke algebra:

$$\mathcal{H} = \mathcal{H}(G, K) = C_c(K \backslash G / K) \text{ with mult.}$$

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1} g)\, dx$$

There is a functor $\qquad$ smooth rep^n of G $\longrightarrow$ $\mathcal{H}$-modules

$(\pi, G, V) \qquad (\pi, \mathcal{H}, V^K)$

action of $\mathcal{H}$ by

$\pi(f) v = \int_G f(g) \pi(g) v\, dg$ $\qquad$ adm. rep. $\longrightarrow$ fin. dim.

unr. rep. $\longrightarrow$ 1-dim.

$\mathcal{H}$ is an imitation of the group algebra.

<u>Satake isomorphism</u>: The following diagram of $\mathbb{C}$-algebras commutes for all unr. rep^n π of G

$$\begin{tikzcd} \delta_B^{-1} 1_{(N,\psi)\lambda K} \quad \mathbb{C}((N,\psi)\backslash G/K) \arrow[r, "\mathcal{F}", "\text{Gasselman Shalika}"'] & \mathcal{H} \arrow[d, "S_{\mathcal{H},\pi}"] \arrow[r, leftrightarrow, "\sim"] & \mathrm{Rep}(GL_n(\mathbb{C})) \arrow[dl, "S_{Rep,\pi}"] \\ & \mathbb{C} & \end{tikzcd}$$

$A_\lambda \qquad V_\lambda$

We identify weights of $GL_n(\mathbb{C})$ $\updownarrow$ coweights of $GL_n(k)$

where $S_{Rep,\pi}(\tau, W) = tr_W(\tau(t_\pi))$

$v^\circ \in V^K$, $v^\circ \neq 0$

$S_{\mathcal{H}}(f)\, v^\circ = \int f(g)\pi(g)v^\circ\, dg \leftarrow$ Spectral decomposition of $\mathcal{H}$

$f^{\psi,N} = \mathcal{F}_\psi(f) = \int_N f(ng)\psi(n)\,dn$

<u>Example (Jacquet-Langlands)</u>: $G = GL_2$, $N = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$

Let (π, V) be an unr. rep^n of G.

<u>Fact</u>: $\dim \mathrm{Hom}_N(\pi, \mathbb{C}_\psi) \geq 1$ for a nondegenerate additive char. ψ of k. (In fact, this is an equality)

Assume ψ is of conductor $\mathcal{O}$

Take $\Lambda \in \mathrm{Hom}_N(\pi, \mathbb{C}_\psi)$ s.t. $\Lambda(v^\circ) = 1$, and let

$\rightarrow$ $W_{v,\psi}(g) := \Lambda(\pi(g)v)$ for $v \in V$ and let $W = W_{v^\circ,\psi}$ (unig. for an unr. rep^n)

We would like to show: $\int_{k^\times} W\begin{pmatrix}a & \\ & 1\end{pmatrix}|a|^{s-\frac{1}{2}}\, d^\times a = L(s,\pi,st)$

<u>Def</u>: Let (ρ, W) be a f. dim. rep^n of $GL_n(\mathbb{C})$, we call $\Delta(g,s) \in \mathcal{H}[q^{-s}]$ a <u>generating function</u> for $L(s, \cdot, \rho)$ if for all smooth irr. unr. rep^n π of G and any $l \in V^\vee$ we have

$$\int_G l(\pi(g)v^\circ)\Delta(g,s)\,dg = L(s,\pi,\rho)\; l(v^\circ)$$

<u>Prop</u>: $\exists!\ \Delta(g,s)$

<u>Proof</u>: Uniqueness follows from the spectral decomposition of $\mathcal{H}$.

<u>Existence</u>: We have Poincaré identity $\rho: GL_n(\mathbb{C}) \to GL_N(\mathbb{C})$

Choose t_π st $\rho(t_\pi)$ is diagonal

$$L(s,\pi,\rho) = \frac{1}{\det(1 - q^{-s}\rho(t_\pi))} = \prod_{i=1}^N \frac{1}{1 - q^{-s}\rho(t_\pi)_{ii}} = \prod_{i=1}^N \sum_{k=0}^\infty (q^{-s}\rho(t_\pi)_{ii})^k =$$

$$= \sum_{k=0}^\infty \Bigg(\sum_{\substack{i_1,\dots,i_N \\ i_1+\dots+i_N = k}} \prod_{j=1}^N \rho(t_\pi)_{i_j i_j} \Bigg) q^{-ks} = \sum_{k=0}^\infty tr_{Sym^k \rho}(\rho(t_\pi))\, q^{-ks}$$

Remark: $\forall\, l \in V^\vee$ and $f \in \mathcal{H}$ we have

$$\int_G f(g)\, l(\pi(g) v^\circ)\, dg = l\left(\int_G f(g)\pi(g) v^\circ\, dg\right) = l\left(S_{\mathcal{H},\pi}(f) v^\circ\right) = S_{\mathcal{H},\pi}(f)\, l(v^\circ)$$

Now, let λ_k be the highest weight of $\mathrm{Sym}^k \rho$ and let $A_{\lambda_k} = A_k$ be the corresponding function in $\mathcal{H}$, define

$$\Delta(g,s) = \sum_{k=0}^{\infty} A_k\, q^{-ks}$$

$\square$

This comes with a price, A_λ are hard to describe

<u>Back to J-L:</u>

$$L(s,\pi,\rho) = \int_G \Delta(g,s)\, \Lambda(\pi(g) v^\circ)\, dg = \int_G \Delta(g,s)\, W(g)\, dg =$$

$$= \int_{N\backslash G} \int_N \Delta(ng,s)\, W(ng)\, dn\, dg = \int_{N\backslash G} \left(\int_N \Delta(ng,s)\, \psi(n)\, dn \right) W(g)\, dg =$$

$$= \int_{N\backslash G} \Delta^{\psi,N}(g,s)\, W(g)\, dg \underset{\uparrow}{=} \int_T \Delta^{\psi,N}(t)\, W(t)\, dt$$

$N\backslash G / K = T / T(\mathcal{O})$ right K inv.

On the other hand $\Delta^{\psi,N} = \sum_{k=0}^{\infty} A_k^{\psi,N}(t)\, q^{-ks} = \sum_{k=0}^{\infty} \delta_B^{-1/2}(t)\, \mathbb{1}_{(N,\psi)\lambda_k K}(t)\, q^{-ks} =$

$$= \sum_{k=0}^{\infty} \delta_B^{-1/2}(t)\, \mathbb{1}_{(N,\psi)\begin{pmatrix}\varpi^{-k} & \\ & 1\end{pmatrix} K}(t)\, q^{-ks}$$

$$\Delta^{\psi,N}\begin{pmatrix} a & \\ & 1\end{pmatrix} = \sum_{k=0}^{\infty} \mathbb{1}_{\varpi^{-k}\mathcal{O}^\times}(a)\, |a|^{s-1/2}$$

And thus

$$\int_T \Delta^{\psi,N}(t)\, W(t)\, dt = \int_{k^\times} W\begin{pmatrix} a & \\ & 1\end{pmatrix} |a|^{s-1/2}\, d^\times a = L(s,\pi,\rho)$$

What happens for other groups?

Let G be a reductive algebraic group G/k

__LLC:__

Packets of smooth irr. rep^n of $G(k)$ $\longleftrightarrow$ gp. hom. $W_F' \longrightarrow {}^LG(\mathbb{C})$

Take $U \leq G$ a unipotent subgroup with a nontrivial character ψ. We have

$$C((N,\psi)\backslash G/k) \xleftarrow{\ \mathcal{F}\ } \mathcal{H}(G,k) \simeq Rep({}^LG(\mathbb{C}))$$

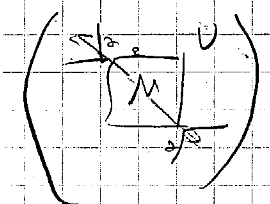
$f \mapsto \int_k f(t g)\,dk$

$$\searrow \mathbb{C} \swarrow$$

$\mathcal{F}$ is no longer an isomorphism and we don't have a formula for $A_\lambda^{\psi,U}$. For arbitrary G and π, π might not support an ψ,N F.C. i.e. $\dim_N(\pi,\mathbb{C}_\psi)=0$. We do know that $\dim_U(\pi,\mathbb{C}_\psi)>0$ for some U and ψ.

__Example__ $G = G_2(k) = Aut(\mathbb{O}(k))$ $\overset{\alpha}{\bullet} \Lleftarrow \overset{\beta}{\bullet}$ ${}^LG^2(\mathbb{C})$

$G_2 \hookrightarrow SO(8), SO(7)$ s.t. $G_2(\mathbb{C}) \hookrightarrow SO_7(\mathbb{C})$ min. faithfull rep^n



$P = MU$ $M \cong GL_2$, U - Heisenberg group of dim 5

ψ - certain complex char. of U.

Let π be an irr. unr. rep^n of G that support a U,ψ F.C.

__Conj:__ $\forall\, l \in Hom_U(\pi, \mathbb{C}_\psi)$ ($\neq 0$ by assumption)

We have $\int_{U\backslash G} F_s(g)\, l(\pi(g)v^0)\,dg = L(s,\pi,st)\,l(v^0)$

where F_s is given by global considerations.

__How to prove this?__

We still have $\Delta(\cdot,s) \in \mathcal{H}[[q^{-s}]]$ s.t. $\forall l \in \pi^\vee$ $\int_G l(\pi(g)v^0)\Delta(g,s)\,dg = L(s,\pi,st)\,l(v^0)$

Taking $l \in Hom_U(\pi,\mathbb{C}_\psi)$ s.t. $l(v^0)=1$ we have

$$\int_{U\backslash G} \Delta^{\psi,U}(g,s)\; l(\pi(g)v^0)\,dg = L(s,\pi,st)$$

So we want to prove $\Delta^{\psi,U}(g,s) = F_s(g)$

But we don't know to write down a formula for $\Delta(g,s)$ or $\Delta^{\psi,U}(g,s)$

The "obvious" thing is to copy the formula of $\Delta(g,s)$ for other groups. for
$$t = \begin{pmatrix} a & & & & \\ & b/a & & & \\ & & a^2/b & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}$$
define $D(t,s) = |a|^{s-3}$

Fix unr. vec: $v^{\circ\vee} \in V^\vee$ s.t. $\langle v^{\circ\vee}, v^{\circ} \rangle = 1$ and let $s_\pi(g) = \langle \pi(g) v^{\circ}, v^{\circ\vee} \rangle$ ← called spherical function of π

$s_\pi(g)$ has a nice formula so we can compute

$$\int_G D_s(g)\, s_\pi(g)\, dg = \frac{P_1(q^{-s}) - P_2(q^{-s})\, tr_{st}(t_\pi)}{N(q^{-s})} L(s,\pi,st)$$

Where P_1, P_2 are poly. and N is a rational function all ind. of π.

spectral decomposition $\Longrightarrow D_s = \Delta(\cdot,s) * P_s$ where $P_s = \dfrac{P_1(q^{-s}) A_0 - P_2(q^{-s}) A_1}{N(q^{-s})}$

Now we need to prove two things:

1) $\Delta^{\psi,U}_s * P_s = D_s^{\psi,U} \overset{?}{=} F_s * P_s$

2) $*P_s$ is invertible.

Proof for 2): $\mathcal{H}$ can be complited into a C^*-algebra

Where A_0 is the unit of $\hat{\mathcal{H}}$

We need to prove $A_0 - \frac{P_2}{P_1} A_1$ is inv.

It is enough to show that $\left\| \dfrac{P_2(q^{-s})}{P_1(q^{-s})} A_1 \right\| < 1$

$$\| \qquad \left| \frac{P_2(q^{-s})}{P_1(q^{-s})} \right| \|A_1\|$$

I don't know to compute $\|A_1\|$ but I do know $\exists s_0\ \forall\, \mathrm{Re}(s) > s_0$ $\left| \dfrac{P_2(q^{-s})}{P_1(q^{-s})} \right| < \|A_1\|$