L-functions & their generating functions

Let $k = \mathbb{Q}_p$

- $\mathcal{O}$ the ring of integers of k, (p) the maximal ideal in $\mathcal{O}$

Let $G = GL_2(k)$, $K = GL_2(\mathcal{O})$ (a max. cpt. subgp.)

$T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

A rep. (π, V) of G is called smooth if for any $v \in V$

$\mathrm{stab}_G(v)$ is open in G

(π, V) is called <u>unramified</u> if it is smooth, irr. and $\dim V^K = 1$

<u>Satake iso. I</u>: There is a correspondence

$$\left\{ \begin{array}{l} \text{Unr. rep} \\ \text{of } G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{s.s. conj} \\ \text{classes of } GL_2(\mathbb{C}) \end{array} \right\} \qquad {}^L GL_n(k) = GL_n(\mathbb{C})$$

π $\qquad t_\pi$ - Satake parameter of π $\qquad {}^L G$

$t_\pi = \begin{pmatrix} \eta_1 & \\ & \eta_2 \end{pmatrix} \longrightarrow \chi : T \to \mathbb{C}^\times \quad \chi\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} = |t_1|^{\eta_1} |t_2|^{\eta_2} \rightsquigarrow$ Extend χ to B

$\rightsquigarrow \pi$ = unique unr. subquotient of $\mathrm{Ind}_B^G \chi$

<u>Def</u>: For an unr. rep π of G, and a f. dim. rep ρ of $GL_2(\mathbb{C})$

let $$L(s, \pi, \rho) = \frac{1}{\det(1 - \rho(t_\pi) q^{-s})}$$

<u>Goal</u>: Write $L(s, \pi, \rho)$ as an integral.

<u>Motivation</u>: ① <u>G-S</u>: Give a definition of L-functions for ramified rep.

② L-functions encode information about π that might

be more accessible via the integral rep.

③ Global reasons. <u>Ex</u>: mero. cont.

Note that $\mathrm{Hom}(N, \mathbb{C}^\times) = \mathrm{Hom}(k, \mathbb{C}^\times)$, fix non trivial $\psi \in \mathrm{Hom}(N, \mathbb{C}^\times)$

Won't be used <u>Fact</u>: $\dim \mathrm{Hom}_N(\pi, \mathbb{C}_\psi) = 1$ (uniq. of Whittaker model)]

<u>Thm</u>: (J-L, Hecke)

For <u>any</u> unr. (π, V) and any $l \in \mathrm{Hom}_N(\pi, \mathbb{C}_\psi)$

$$\int_{k^\times} l\left(\pi \begin{pmatrix} a & \\ & 1 \end{pmatrix} v^\circ\right) |a|^{s + \frac{1}{2}} \, d^\times a = L(s, \pi, \mathrm{st})\, l(v^\circ)$$

Spherical Hecke algebra: $\mathcal{H} = \mathcal{H}(G,K) = C_c(K\backslash G/K)$ [(unitary [Iw] for p groups]

with mult. $(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g)\,dx$

There is a functor: smooth G-rep. $\longrightarrow$ $\mathcal{H}$-modules

$(\pi, G, V) \longmapsto (\pi, \mathcal{H}, V^K)$

$\pi(f)v = \underbrace{\int_G f(g)\pi(g)v\,dg}_{\text{finite sum}}$

unr. rep. $\longmapsto$ 1-dim modules $\leftarrow$ $\mathcal{H}$ acts by char.

Satake iso. II: There is a spectral decomposition of $\mathcal{H}$ by unr. rep. such that $\forall \pi$ the following diagram commutes

$$\begin{tikzcd} \overset{A_\lambda}{\mathcal{H}} \arrow[r, "\sim"] \arrow[dr, "S_\pi"'] & \overset{V_\lambda \ \text{Basis}}{\mathrm{Rep}(GL_2(\mathbb{C}))} \arrow[d, "S_\pi"] \\ & \mathbb{C} \end{tikzcd}$$

$S_\pi(f) = \pi(f)|_{V^K}$

$S_\pi(\rho, W) = \mathrm{tr}_W\, \rho(t_\pi)$

Basis of $\mathcal{H}$: ① $1_{K\gamma K}$ ② $A_\lambda = ?$

Def: We call $\Delta(g,s) \in \mathcal{H}[[q^{-s}]]$ a generating function for $L(s,\pi,\rho)$ if for any unr. rep. π of G and any $l \in \pi^\vee$ we have

$$\int_G l(g\cdot v^\circ)\Delta(g,s)\,dg = L(s,\pi,\rho)\, l(v^\circ) \qquad (0 \neq v^\circ \in V^K)$$

Prop: $\exists!\ \Delta(g,s)$

Proof: $N = \dim \rho$

$L(s,\pi,\rho) = \det(1 - q^{-s}\rho(t_\pi))^{-1} = \prod_{i=1}^{N} (1 - q^{-s}\rho(t_\pi)_{ii})^{-1} = \prod_{i=1}^{N}\sum_{k=0}^{\infty} (q^{-s}\rho(t_\pi)_{ii})^k$

$= \sum_{k=0}^{\infty} \mathrm{tr}(\mathrm{Sym}^k \rho(t_\pi))\, q^{-ks} = \sum_{k=0}^{\infty} S_\pi(\mathrm{Sym}^k\rho)\, q^{-ks} =$

$= \sum_{k=0}^{\infty} S_\pi(A_{k,\rho}\, q^{-ks}) = S_\pi\big(\underbrace{\sum_{k=0}^{\infty} A_{k,\rho}\, q^{-ks}}_{\Delta(g,s)}\big) = \frac{1}{l(v^\circ)}\, l\big(\int_G \Delta(g,s)\pi(g)v^\circ\,dg\big) =$

$= \frac{1}{l(v^\circ)} \int_G \Delta(g,s)\, l(\pi(g)v^\circ)\,dg.$

In order to prove the Thm we need: Denote $A_k := A_{k,st}$

Thm [Casselman-Shalika] $A_k^{\psi,N} = \delta_B^{1/2}\, 1_{(N,\psi)\left(\begin{smallmatrix} \varpi^n & \\ & 1\end{smallmatrix}\right)K}$

Proof of Thm: $L(s,\pi,st) = \int_G \Delta(g,s)\, l(g\cdot v^\circ)\,dg = \int_{N\backslash G}\int_N \Delta(ng,s)\, l(ng\cdot v^\circ)\,dg =$

$= \int_{N\backslash G}\int_N \Delta(ng,s)\psi(n)\,dn\; l(g\cdot v^\circ)\,dg = \int_{N\backslash G} \Delta^{\psi,N}(g,s)\, l(g\cdot v^\circ)\,dg =$

$= \int_T \Delta^{\psi,N}(t,s)\, l(t\cdot v^\circ)\,\delta_B^{-1}(t)\,dt = \sum_{k=0}^{\infty} q^{-ks} \int_T A_k^{\psi,N}(t)\,\delta_B^{-1}(t)\, l(t\cdot v^\circ)\,dt =$

$= \sum_{k=0}^{\infty} q^{-ks} \int_T 1_{(N,\psi)\left(\begin{smallmatrix} \varpi^n & \\ & 1\end{smallmatrix}\right)K}(t)\,\delta_B^{-1/2}(t)\, l(t\cdot v^\circ)\,dt = \sum_{k=0}^{\infty}\int_{|a| = q^{-k}} l\left(\begin{pmatrix} a & \\ & 1\end{pmatrix} v^\circ\right) |a|^{s-\frac12}\,da =$

$= \int_{F^\times} l\left(\begin{pmatrix} a & \\ & 1\end{pmatrix} v^\circ\right) |a|^{s-\frac12}\,da$

$\square$