ζ- and $\mathcal{L}$-functions

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Riemann ζ-function

Definition (Riemann ζ-function)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \downarrow\downarrow \Re e(s) > 1.$$

Theorem (Euler Product, 1740)

$$\begin{aligned} \zeta(s) &= \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots\right) \\ &= \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}. \end{aligned} \qquad \Re e(s) > 1$$

Corollary

There are infinitely many primes $\Leftrightarrow \lim_{s \to 1} \zeta(s) = \infty$.

Prime Number Theorem (PNT)

The connection of $\zeta(s)$ to prime numbers is much deeper:

Prime Number Theorem (Hadamard, de la Vallée Poussin, 1896)

Let $\pi(x) = \#\{p \leq x \mid p \text{ prime}\}$. Then

$$\pi(x) \sim \frac{x}{\ln(x)},$$

i.e. $\lim_{x\to\infty} \frac{\pi(x)\ln(x)}{x} = 1$.

A key fact in analytic proofs of PNT

$$\lim_{x\to 1} \zeta(x+iy) \neq 0 \quad \forall 0 \neq y \in \mathbb{R}.$$

(Wiener, 1951) This is, in fact, equivalent to PNT.

Error Bounds in PNT

Logarithmic Integral

The Europian/Eulerian logarithmic integral is given by

$$Li(x) = \int_2^x \frac{dt}{\ln(t)} \sim \frac{x}{\ln(x)}.$$

A variant of PNT states that $\pi(x) \sim Li(x)$.

How good is the approximation in PNT?

- A natural question is to ask how big could $\left|\pi(x) - Li(x)\right|$ be?
- (Ingham, 1932): $\pi(x) = Li(x) + O\left(x^\beta \ln(x)\right)$ for some $\frac{1}{2} \leq \beta \leq 1$.
- So the best one can hope for is that $\pi(x) = Li(x) + O\left(\sqrt{x}\ln(x)\right)$

Meromorphic Continuation (MC) and Functional Equation (FE)

Theorem (Riemann, 1859)

MC $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ admits a meromorphic continuation to $\mathbb{C}$.

FE Let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then

$$\xi(s) = \xi(1-s).$$

The proof is based on Poisson's Summation Formula (PSF)

For a smooth $f : \mathbb{R} \to \mathbb{C}$ of rapid decay

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k),$$

where $\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \lambda} dx$.

Sketch of Proof of MC and FE

- By definition, for $\Re\mathfrak{e}\,(s) > \frac{1}{2}$:

$$\begin{aligned}\xi(2s) &= \pi^{-s}\Gamma(s)\zeta(2s) = \pi^{-s}\left(\int_0^\infty e^{-t}t^s\frac{dt}{t}\right)\left(\sum_{n=1}^\infty \frac{1}{n^{2s}}\right) \\ &= \int_0^\infty \left(\sum_{n=1}^\infty e^{-t}\left(\frac{t}{\pi n^2}\right)^s\right)\frac{dt}{t} \qquad \left\{t \leftrightarrow \frac{t}{\pi n^2}\right\} \\ &= \int_0^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2 t}\right)t^s\frac{dt}{t}.\end{aligned}$$

- For $\Im\mathfrak{m}\,(z) > 0$, the Jacobi θ-function is defined by

$$\theta(z) = \sum_{n=1}^{\infty} e^{\pi i n^2 z} + \frac{1}{2} = \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{\pi i n^2 z}.$$

$$\begin{aligned}\xi(2s) &= \int_0^\infty \left(\theta(it) - \tfrac{1}{2}\right)t^s\frac{dt}{t} \\ &= \underbrace{\int_1^\infty \left(\theta(it) - \frac{1}{2}\right)t^s\frac{dt}{t}}_{\text{converges for any } s} + \int_0^1 \left(\theta(it) - \frac{1}{2}\right)t^s\frac{dt}{t}\end{aligned}$$

$$\xi(2s) = \int_1^{\infty} \left(\theta(it) - \frac{1}{2}\right) t^s \frac{dt}{t} + \int_0^1 \left(\theta(it) - \frac{1}{2}\right) t^s \frac{dt}{t}$$

By PSF, one gets $\theta(it) = \frac{1}{\sqrt{t}}\theta\left(\frac{i}{t}\right)$ for $t > 0$. Hence,

$$\begin{aligned}
\int_0^1 \left(\theta(it) - \tfrac{1}{2}\right) t^s \tfrac{dt}{t} &= \int_0^1 \theta(it)\, t^s \tfrac{dt}{t} - \tfrac{1}{2s} \\
&\overset{(PSF)}{=} \int_0^1 \theta\left(\tfrac{i}{t}\right) t^{s-\frac{1}{2}} \tfrac{dt}{t} - \tfrac{1}{2s} \\
&= \int_0^1 \left(\theta\left(\tfrac{i}{t}\right) - \tfrac{1}{2}\right) t^{s-\frac{1}{2}} \tfrac{dt}{t} + \tfrac{1}{1-2s} - \tfrac{1}{2s} \qquad \left\{t \leftrightarrow \tfrac{1}{t}\right\} \\
&= \int_1^{\infty} \left(\theta(it) - \tfrac{1}{2}\right) t^{\frac{1}{2}-s} \tfrac{dt}{t} + \tfrac{1}{1-2s} - \tfrac{1}{2s}.
\end{aligned}$$

$$\boxed{\xi(2s) = \underbrace{\int_1^{\infty} \left(\theta(it) - \frac{1}{2}\right)\left(t^s + t^{\frac{1}{2}-s}\right) \frac{dt}{t}}_{\text{entire function}} - \frac{1}{2s(1-2s)}.}$$

We get MC and FE ($2s \leftrightarrow 1 - 2s$) from RHS.

Poles and Zeroes of $\zeta(s)$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{FE}$$

Poles and Zeroes of $\zeta(s)$

- From FE it follows that $\zeta(s)$ is holomorphic for all $s \neq 1$ and admits a simple pole at $s = 1$.
- $\zeta(s)$ is non-zero for $\Re\mathfrak{e}(s) > 1$ (Euler's product formula).
- **Trivial zeroes:** From FE it follows that $\zeta(-2n) = 0$ for $n \geq 1$.
- Any other zero of $\zeta(s)$ must lie in the **critical strip** $0 < \Re\mathfrak{e}(s) < 1$.

No real zeroes in the critical strips: $\zeta(s) \neq 0$ for $0 < s < 1$

Note that for $\Re\mathfrak{e}(s) > 1$:

$$\left(1 - \frac{2}{2^s}\right)\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

The RHS converges for $\Re\mathfrak{e}(s) > 0$ and is non-zero for $0 < s < 1$.

Riemann Hypothesis

Riemann Hypothesis (RH)

If $\zeta(s) = 0$ then $s = -2n$ for $n \geq 1$ or $\Re\mathfrak{e}(s) = \frac{1}{2}$.

No questions about proofs of RH please!

RH $\Leftrightarrow$ Best possible error in PNT

(Von Koch, 1901): RH is equivalent to $\pi(x) = Li(x) + O\left(\sqrt{x}\log(x)\right)$.

There are many other ways to express this idea...

See: `https://mathoverflow.net/questions/39944/collection-of-equivalent-forms-of-riemann-hypothesis`

Dynamical approach to RH

Hilbert-Pólya Conjecture

Let

$$\tau = \left\{ t \in \mathbb{C} \,\middle|\, \zeta\left(\frac{1}{2} + it\right) = 0 \text{ is a non-trivial zero} \right\},$$

then τ is the set of eigenvalues of an unbounded Hermitian operator $\mathcal{H}$ on some Hilbert space .

Berry-Keating

The Riemann operator $\mathcal{H}$ is a quantization of the classical Hamiltonian XP (here P is momentum and X is position).
Furthermore, the classical system associated to $\mathcal{H}$ is chaotic and unstable.

This eigenvalues can be interpreted as frequencies of harmonics,one can actually listen to the music of the prime numbers:
`https://www.youtube.com/watch?v=i1FqnfrcWA4`

Tate's Thesis, 1950

Better Understanding the Completed ζ-function

- $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ satisfy $\xi(s) = \xi(1-s)$, while $\zeta(s)$ satisfy
$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$
- Is there a natural way in which $\xi(s)$ arises?
- For $\mathfrak{Re}(s) > 1$, write

$$\begin{aligned}
\xi(s) &= \pi^{-s/2}\Gamma(s/2)\zeta(s) \\
&= \pi^{-s/2}\Gamma(s/2)\prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \\
&= \prod_{p \leq \infty \text{ prime}} \zeta_p(s),
\end{aligned}$$

where $\zeta_p(s) = \frac{1}{1-p^{-s}}$ for $p < \infty$ and $\zeta_\infty(s) = \pi^{-s/2}\Gamma(s/2)$.

Completions of $\mathbb{Q}$

Theorem (Ostrowski, 1916)

Up to equivalence, any norm on $\mathbb{Q}$ is either the usual absolute value (denoted $|\cdot|_\infty$) or a p-adic norm ($p < \infty$ prime):

$$\left|\frac{m}{n}p^l\right|_p = p^{-l} \quad (m,p) = (n,p) = 1.$$

The Field of p-adic Numbers

The completion of $\mathbb{Q}$ with respect to p is denoted by $\mathbb{Q}_p$ (notation: $\mathbb{Q}_\infty = \mathbb{R}$). This is a locally compact field and for $p < \infty$ it is totally disconnected.
The ring of integers

$$\mathbb{Z}_p = \left\{x \in \mathbb{Q}_p \,\middle|\, \left|x_p\right| \leq 1\right\}$$

is a local ring with maximal ideal (p).

Local Integral Representations

- For $p < \infty$, let $\Phi_p(x) = \mathbf{1}_{\mathbb{Z}_p} \in C_c^\infty(\mathbb{Q}_p)$. One checks that

$$\begin{aligned}\int_{\mathbb{Q}_p^\times} \Phi_p(x)\,|x|_p^s\,d^\times x &= \int_{\mathbb{Z}_p\setminus\{0\}} |x|_p^s\,d^\times x \\ &= \sum_{n=0}^{\infty} \int_{p^n\mathbb{Z}_p\setminus p^{n+1}\mathbb{Z}_p} |x|^s\,d^\times x \\ &= \sum_{n=0}^{\infty} p^{-ns} = \frac{1}{1-p^{-s}} = \zeta_p(s) \quad \Re\mathfrak{e}(s) > 0.\end{aligned}$$

- Similarly, for $\Phi_\infty(x) = e^{-\pi x^2}$, we get

$$\int_{\mathbb{R}^\times} \Phi_\infty(x)\,|x|_\infty^s\,d^\times x = \zeta_\infty(s) \quad \Re\mathfrak{e}(s) > 0.$$

- Note that Φ_p are all eigenfunctions of the Fourier transform on $\mathbb{Q}_p$ (with eigenvalue 1). That would yield a local FE.

- We would like to have a "natural" measure space X with a "good" family of measurable functions f_s on X, which yields

$$\int_X f_s(x)\,dx = \xi(s) = \prod_{\substack{p\leq\infty \\ \text{prime}}} \underbrace{\int_{\mathbb{Q}_p^\times} \Phi_p\left(x_p\right)\left|x_p\right|_p^s d^\times x_p}_{\zeta_p(s)} \quad \Re\mathfrak{e}\,(s) > 1.$$

  and that $\int_X f_s(x)\,dx$ gives rise to MC and FE on $\mathbb{C}$.
- The ring of adeles is the restricted product of all $\mathbb{Q}_p$:

$$\mathbb{A} = \prod_{\substack{p\leq\infty \\ \text{prime}}}{}' \mathbb{Q}_p = \left\{\left(x_p\right)_{p\leq\infty} \,\middle|\, x_p \in \mathbb{Z}_p \text{ for a.a. } p\right\}.$$

- This is a locally compact (the direct product isn't!) normed ring (generate a topology finer than the product topology) and $\mathbb{Q}\backslash\mathbb{A}$ is compact.
- Taking $X = \mathbb{A}^\times$ and $f_s\left(\left(x_p\right)_{p\leq\infty}\right) = \prod_{p\leq\infty} \Phi_p\left(x_p\right)\left|x_p\right|_p^s$ works!

Dirichlet $\mathcal{L}$-functions

Dirichlet $\mathcal{L}$-functions

Let $\chi : \mathbb{Z} \to \mathbb{C}$ be a strongly multiplicative periodic function, i.e.

$$\chi(m \cdot n) = \chi(m)\chi(n) \quad \forall m, n \in \mathbb{Z}$$
$$\exists d \in \mathbb{N} : \chi(m + d) = \chi(m) \, \forall m \in \mathbb{Z}.$$

Let

$$\mathcal{L}^{\{\infty\}}(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p < \infty \text{ prime}} \underbrace{\frac{1}{1 - \chi(p) p^{-s}}}_{\mathcal{L}_p(\chi, s)} \qquad \Re e(s) > 1$$

Note that $\mathcal{L}^{\{\infty\}}(\mathbf{1}, s) = \zeta(s)$.

The fact that $\mathcal{L}(\chi, 1) \in \mathbb{C}^\times$ for any $\chi \neq \mathbf{1}$ implies:

Dirichlet's Theorem on Arithmetic Progressions (1837)

If $(a, b) = 1$, then the arithmetic progression $c_n = a \cdot n + b$ admits infinitely many primes.

- One can define $\mathcal{L}_\infty(\chi, s)$ and $\mathcal{L}(\chi, s) = \mathcal{L}^{\{\infty\}}(\chi, s)\,\mathcal{L}_\infty(\chi, s)$ so that $\mathcal{L}(\chi, s)$ can be expressed as an integral of the form:

$$\int_{\mathbb{A}^\times} \Phi(x)\,\widetilde{\chi}(x)\,|x|^s\,d^\times x,$$

where $\widetilde{\chi} : \mathbb{A}^\times \to \mathbb{C}^\times$ is a "grössencharacter" associated with the strongly multiplicative periodic function $\chi : \mathbb{Z} \to \mathbb{C}^\times$.
- MC (and a FE) of $\mathcal{L}(\chi, s)$ follows.
- Note that $\mathbb{A}^\times = GL_1(\mathbb{A})$ and χ an irreducible representation. We want to generalize the notion of an $\mathcal{L}$-function to representations of other "adelic groups".

Generalization to Other Groups

- Let G be a split reductive group defined over $\mathbb{Z}$ (so a functor $G : \mathbb{Z}\text{-Algebras} \to \text{Groups}$).
- To G we can associate a complex Lie group $G^{\vee}$ which helps to encode information about representation of $G\left(\mathbb{Q}_p\right)$.
- For example: $GL_n^{\vee} = GL_n(\mathbb{C})$, $Sp_{2n}^{\vee} = SO_{2n+1}(\mathbb{C})$, $SO_{2n+1}^{\vee} = Sp_{2n}(\mathbb{C})$.
- For any irreducible representation π of $G(\mathbb{A})$ we can define, for almost all primes p, a semi-simple conjugacy class $t_{\pi,p} \in G^{\vee}$.

Langlands $\mathcal{L}$-functions

Langlands $\mathcal{L}$-functions

Let

- $\rho : G^{\vee} \to GL_N(\mathbb{C})$ be a finite-dimensional representation.
- π be an irreducible representation of $G(\mathbb{A})$.
- S be a finite set of primes such that $t_{\pi,p}$ is defined for all $p \notin S$.

The partial Langlands $\mathcal{L}$-function of π with respect to ρ is given by

$$\mathcal{L}^S(\pi, s, \rho) = \prod_{p \notin S} det\left(\mathbf{1}_N - \rho\left(t_{\pi,p}\right) p^{-s}\right)^{-1} \downarrow\downarrow \quad \Re\mathfrak{e}(s) \gg 0.$$

Dirichlet $\mathcal{L}$-functions when: $G = GL_1$, $\rho = id$ and π is 1-dimensional.

Conjecture (Langlands)

If π is a cuspidal automorphic representation, then $\mathcal{L}^S(\pi, s, \rho)$ has MC.

How can this be proven?

The Rankin-Selberg Method

- The main technique to study automorphic $\mathcal{L}$-functions is by integral representations.
- A simple prototype:

$$\mathcal{Z}(\varphi, E_s) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi(g) E_s(g) \ dg,$$

where $\varphi \in \pi$ and E_s is a restriction of a meromorphic family (in the variable s) of automorphic forms (in the variable g) on a bigger group.
- One wants to show $\mathcal{Z}(\varphi, E_s) = \mathcal{L}^S(\pi, s, \rho)\, \mathcal{Z}_S(\varphi, E_s)$, and that $\mathcal{Z}_S(\varphi, E_s)$ is entire and non-vanishing.
- The convergence is guaranteed by the cuspidality of π. MC follows from that of $E_s(g)$.
- Added benefit: $\left\{\text{poles of } \mathcal{L}^S(\pi, s, \rho)\right\} \subset \{\text{poles of } E_s(g)\}$.

When is this conjecture known (for π cuspidal)?

Partial list of known integral representations:

- Hecke, Jacquet-Langlands, Godement-Jacquet:
  For $G = GL_n$ $(G^\vee = GL_n(\mathbb{C}))$ and $\rho = id : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$.
- Jacquet, Piatetski-Shapiro, Shalika:
  $G = GL_n \times GL_m$ $(G^\vee = GL_n(\mathbb{C}) \times GL_m(\mathbb{C}))$ with
  $\rho = \otimes : GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \to GL_{m \cdot n}(\mathbb{C})$ the tensor product.
- There are other sporadic examples involving GL_n with n small.
- Piatetski-Shapiro, Rallis:
  For G classical, ($G^\vee$ is a complex classical group) and ρ the standard representation.
- Ginzburg-Hundley, Gurevich-S. :
  The exceptional group of type G_2 $(G_2^\vee = G_2(\mathbb{C}))$ ρ standard.

What are Automorphic $\mathcal{L}$-functions Good For?

- $\mathcal{L}^S(\pi, s, \rho)$ plays an important role in the Langlands program and the study of their poles is an important ingredient in studying instances of functoriality.
- In particular, the analytic behavior, and in particular the poles and zeros, of $\mathcal{L}^S(\pi, s, \rho)$ parametrize cuspidal representations in terms of functorial lifts.
- MC of all symmetric power $\mathcal{L}$-functions of GL_n would imply the generalized Ramanujan conjecture.
- The analytic behavior of the tensor product $\mathcal{L}$-functions on $G = GL_n \times GL_m$ can determine whether π admits a realization as a cuspidal representation via the converse theorem (If $\mathcal{L}(\pi \otimes \tau, s, \otimes)$ behaves "as it should" for all τ, then π admits such a realization).
- $\mathcal{L}$-functions of automorphic forms on GL_2 are part of the proof of Fermat's last theorem (see Langlands-Tunnell theorem) and hoped to be part of the proof of RH.

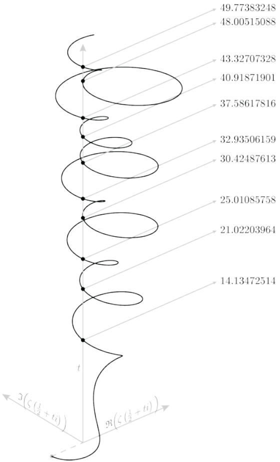
Thank You!

This is a graph of $\zeta\left(\frac{1}{2}+it\right)$ for $0 \le t \le 50$.

The dots are known zeroes of $\zeta\left(\frac{1}{2}+it\right)$.

In fact (Gourdon, Demichel, 2004), all zeroes of $\zeta(s)$ with $\left|\Im\mathfrak{m}(s)\right| < 10^{24}$ are known to be on the critical line.

The number of zeroes in that rectangle has order of magnitude 10^{13}.



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PNT - Sketch of Proof (Due to Newman, Zagier)

- The first Chebyshev function: $\vartheta(x) = \sum_{p \le x} \log(p)$
- Note that $\pi(x) = \sum_{p \le x} 1$.
- Step I: $\vartheta(x) \sim x \Leftrightarrow$ PNT.
- Step II: $\zeta(s) - \frac{1}{s-1}$ has an analytic continuation to $\Re\mathfrak{e}(s) > 0$.
- Step III: $-\frac{\zeta'(s)}{\zeta(s)} = \Phi(s) + \sum_{p \le x} \frac{\log(p)}{p^s(p^s-1)}$, where $\Phi(s) = \sum_p \frac{\log(p)}{p^s}$. It follows that $\Phi(s)$ has analytic continuation to $\Re\mathfrak{e}(s) > \frac{1}{2}$ with poles at $s = 1$ and at zeros of $\zeta(s)$.
- Step IV: $\zeta(1+iy) \neq 0$, with $y \neq 0$, hence $\Phi(s)$ is analytic at $1+iy$.
- Step V: For $\Re\mathfrak{e}(s) > 1$, $\Phi(s) = s\int_0^\infty e^{-st}\vartheta\left(e^t\right)dt$ and hence, by a Tauberian-like argument, $\int_0^\infty \frac{\vartheta(x)-x}{x^2}dx$ converges.
- It follows that $\vartheta(x) \sim x$.