New Way Integrals

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The Ring of Adeles

- To each prime p we associate the field $\mathbb{Q}_p$ of p-adic numbers. In particular $\mathbb{Q}_\infty = \mathbb{R}$.
- We form the restricted product

$$\mathbb{A} = \prod_p{}' \mathbb{Q}_p = \left\{ \left(x_p \right)_p \,\middle|\, x_p \in \mathbb{Z}_p \text{ for almost all } p \right\}.$$

  $\mathbb{A}$ is called *the ring of adeles of* $\mathbb{Q}$.
- This is a normed locally compact ring and $\mathbb{Q}$ embeds (diagonally) into $\mathbb{A}$ as a discrete subring with compact quotient $\mathbb{Q}\backslash\mathbb{A}$. $\mathbb{Q}\backslash\mathbb{A}$ is actually the Pontryagin dual of $\mathbb{Q}$.
- The invertible elements $\mathbb{I} = \mathbb{A}^\times$ are called *ideles*. $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ is not compact but has finite volume.
- For an algebraic group defined over $\mathbb{Z}$ we have an isomorphism between the group of $\mathbb{A}$-points and the restricted product

$$\prod_p{}' G\left(\mathbb{Q}_p\right) \left\{ \left(g_p \right)_p \,\middle|\, g_p \in G\left(\mathbb{Z}_p\right) \text{ for almost all } p \right\}.$$

Automorphic Representations of $GL_2(\mathbb{A})$

- The space of automorphic forms on $GL_2(\mathbb{A})$ is the space $\mathcal{A}$ of functions $GL_2(\mathbb{A}) \to \mathbb{C}$ which are left-$Z(\mathbb{A}) \cdot GL_2(\mathbb{Q})$-invariant with some growth and finiteness conditions. This is a representation of $GL_2(\mathbb{A})$ and its constituents are called *automorphic representations*.
- There is a construction

Cuspidal Hecke Eigenform f	$\rightsquigarrow$	Admissibale representation $\pi_{f,\infty}$ of $GL_2(\mathbb{R})$	$\rightsquigarrow$	Automorphic Representation, π_f of $GL_2(\mathbb{A})$

  where $\pi_{f,\infty}$ and π_f are irreducible representations.
- There representation π_f satisfy the following property

$$CT(\varphi)(g) = \int_{\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} dg\right) dx = 0 \quad \forall g \in GL_2(\mathbb{A}), \ \varphi \in \pi_f.$$

  Representations that satisfy this property are called *cuspidal representations*.

The Partial $\mathcal{L}$ of an Autormorphic Representation

Flath's Theorem

Any irreducible representation π of $GL_2(\mathbb{A})$ is a restricted tensor product $\otimes'_p \pi_p$ where π_p is spherical for almost all p.

Satake Isomorphism (Take I)

There is a correspondence

$$\left\{\begin{matrix}\text{equivalence classes of} \\ \text{irreducible unramified} \\ \text{representations of } GL_2\left(\mathbb{Q}_p\right)\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{semisimple conjugacy} \\ \text{classes in } GL_2(\mathbb{C})\end{matrix}\right\}$$

$$\pi \mapsto t_\pi$$

- Given a finite dimensional representation ρ of ${}^L GL_2 = GL_2(\mathbb{C})$ and an irreducible spherical representation π_p of $GL_2\left(\mathbb{Q}_p\right)$ we define the local $\mathcal{L}$-function to be

$$\mathcal{L}\left(s, \pi_p, \rho\right) = \det\left(\mathbb{1} - p^{-s} \rho\left(t_{\pi_p}\right)\right)^{-1}, \quad s \in \mathbb{C}.$$

- Given an irreducible cuspidal representation $\pi = \otimes'_p \pi_p$ of $GL_2\left(\mathbb{Q}_p\right)$, and a finite set of "bad" places S such that π_p is spherical outside of S, we define the partial $\mathcal{L}$-function of π to be

$$\mathcal{L}^S(s, \pi, \rho) = \prod_{p \notin S} \mathcal{L}\left(s, \pi_p, \rho\right).$$

  This product converges absolutely, to an analytic function, for $Re(s) >> 0$.

Conjecture (Langlands)

$\mathcal{L}^S(s, \pi, \rho)$ admits a meromorphic continuation to $\mathbb{C}$.

Rankin-Selberg Method

1. Cook up a global integral with ”good properties”. Example

$$\mathcal{Z}(s,\varphi) = \int\limits_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)|a|^{s-\frac{1}{2}}\, d^\times a.$$

2. Unfolding + Euler product.
3. Unramified calculation.
4. Deal with the places $p \notin S$.

Theorem(Hecke, Jacquet-Langlands)

For a pure tensor $\varphi = \otimes'_p \varphi_p$, and a finite set of places S such that φ_p is spherical outside S, we have

$$\mathcal{Z}(s,\varphi) = \mathcal{L}^S(s,\pi,\mathfrak{st})\, d(s,\varphi_S),$$

where $d(s,\varphi_S)$ is meromorphic. In particular, $\mathcal{L}(s,\pi,\mathfrak{st})$ admits meromorphic continuation.

Unfolding

- By harmonic analysis: $\varphi(g) = \sum_{\psi \in \widehat{\mathbb{Q}\backslash\mathbb{A}}} \int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \overline{\psi(x)} dx$.
- Fix non-trivial $\psi \in \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$, all other elements in $\widehat{\mathbb{Q}\backslash\mathbb{A}}$ are given by $\psi(\gamma x)$ for $\gamma \in \mathbb{Q}$.
- By cuspidality: $\varphi(g) = \sum_{\gamma \in \mathbb{Q}^\times} \int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \overline{\psi(\gamma x)} dx$.
- The Whittaker functional on π is given by

$$\mathcal{W}_\varphi^\psi(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \overline{\psi(x)} dx.$$

- Note that by automorphicity we have

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \overline{\psi(\gamma x)} dx = \int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right) \overline{\psi(x)} dx$$

- Plugging this into $\mathcal{Z}(s,\varphi)$:

$$\mathcal{Z}(s,\varphi) = \int\limits_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \sum_{\gamma \in \mathbb{Q}^\times} \mathcal{W}_\varphi^\psi \left(\begin{pmatrix} \gamma a & 0 \\ 0 & 1 \end{pmatrix} \right) |a|^{s-\frac{1}{2}} \, d^\times a$$
$$= \int\limits_{\mathbb{A}^\times} \mathcal{W}_\varphi^\psi \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) |a|^{s-\frac{1}{2}} \, d^\times a.$$

- Write $\mathcal{W}^\psi(\varphi) = \mathcal{W}_\varphi^\psi(1)$, we have $\mathcal{W}_\varphi^\psi(g) = \mathcal{W}^\psi(\pi(g)\varphi)$.
- $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{G}_a \right\}$, $\mathcal{W}^\psi \in \mathrm{Hom}_{N(\mathbb{A})}(\pi,\psi) \cong \mathbb{C}$. By the (local and global) uniqueness of Whittaker model we have $\mathcal{W}^\psi = \otimes_p' \mathcal{W}^{\psi_p}$, where $\psi = \otimes_p \psi_p$.

New-Way Integrals

- Given a "nice" variety X and a pure tensor $F = \otimes_p F_p$ we have an Eulerian product

$$\int_{X(\mathbb{A})} F(x)\,dx = \prod_p \int_{X(\mathbb{Q}_p)} F_p\left(x_p\right) dx_p$$

- By the uniqueness property above, for a pure tensor $\varphi = \otimes_p \varphi_p$, we have $\mathcal{W}^{\psi}(\varphi) = \prod_p \mathcal{W}^{\psi_p}\left(\varphi_p\right)$. Hence

$$\int_{\mathbb{A}^\times} \mathcal{W}^{\psi}_{\varphi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}}\, d^\times a = \prod_p \int_{\mathbb{Q}_p^\times} \mathcal{W}^{\psi_p}\left(\pi\left(\begin{pmatrix} a_p & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_p\right) \left|a_p\right|^{s-\frac{1}{2}}\, d^\times a_p$$

- We want to obtain our result without using the uniqueness of the Whittaker functional. We will prove a weaker form of this.

- Instead of using the uniqueness of the Whittaker functional, we use the following local result.

Unramified Computation

Let π_p be unramified and be φ_p be spherical, then **for any** $\ell \in \mathrm{Hom}_{N(\mathbb{Q}_p)}\left(\pi_p, \psi_p\right)$ we have

$$\mathcal{Z}_p\left(s, \varphi_p\right) = \int_{\mathbb{Q}_p^\times} \ell\left(\pi_p\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_p\right)|a|_{\mathbb{Q}_p}^{s-\frac{1}{2}}\, d^\times a = \mathcal{L}\left(s, \pi_p, \mathfrak{st}\right)\ell\left(\varphi_p\right)$$

- Note that when using the unique model, we need to prove this for a unique functional l, however in a general setting the space $\mathrm{Hom}_{N(\mathbb{Q}_p)}\left(\pi_p, \psi_p\right)$ can be infinite dimensional. We will later see why such a claim can still make sense when π_p is unramified.

The Main Theorem from the Unramified Computation

- Roughly speaking:

$$\mathcal{Z}(s,\varphi) = \lim_{\longrightarrow S\subset\Omega\subset\mathcal{P}\,|\Omega|<\infty} \int_{\mathbb{A}_\Omega^\times} I\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right)|a|^{s-\frac{1}{2}}\, d^\times a$$

- $$\int_{\mathbb{A}_{\Omega\cup\{p\}}^\times} I\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right)|a|^{s-\frac{1}{2}}\, d^\times a \qquad (p\notin\Omega)$$

$$= \int_{\mathbb{A}_\Omega^\times} |a_\Omega|^{s-\frac{1}{2}}\left[\int_{\mathbb{Q}_p^\times} \mathcal{W}^\psi\left(\pi\left(\begin{pmatrix} a_p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_\Omega & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right)\left|a_p\right|^{s-\frac{1}{2}}\, d^\times a_p\right] d^\times a_\Omega$$

$$= \mathcal{L}\left(s,\pi_p,\mathfrak{st}\right)\int_{\mathbb{A}_\Omega^\times} \mathcal{W}^\psi\left(\pi\left(\begin{pmatrix} a_\Omega & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right)|a_\Omega|^{s-\frac{1}{2}}\, d^\times a_\Omega$$

$\mathbb{A}_\Omega^\times = \prod_{p\in\Omega}\mathbb{Q}_p^\times$

Writing

$$d(s,\varphi_S) = \int_{\mathbb{A}_S^\times} \mathcal{W}^\psi\left(\pi\left(\begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right)|a_S|^{s-\frac{1}{2}}\, d^\times a_S$$

we have (for $\mathfrak{Re}(s) >> 0$)

$$\mathcal{Z}(s,\varphi) = \lim_{\substack{\longrightarrow \\ S\subset\Omega\subset\mathcal{P} \\ |\Omega|<\infty}} d(s,\varphi_S)\prod_{p\in\Omega\setminus S}\mathcal{L}\left(s,\pi_p,\mathfrak{st}\right) = d(s,\varphi_S)\,\mathcal{L}^S(s,\pi,\mathfrak{st})\,.$$

We shall not prove the statement regarding $d(s,\varphi_S)$, we shall discuss the unramified computation. We fix a prime p such that π_p is unramified and φ_p is spherical.

Satake Isomorphism - Take II

For any unramified irreducible representation (π, V) we have a c.d.:

$$\begin{array}{ccc} \mathcal{H}\left(GL_2\left(\mathbb{Q}_p\right), GL_2\left(\mathbb{Z}_p\right)\right) & \overset{\mathfrak{s}}{\longleftrightarrow} & \mathbb{C}\cdot Rep\left(GL_2\left(\mathbb{C}\right)\right) . \\ & {\scriptstyle \mathfrak{s}_{Hecke,\pi}} \searrow \quad \swarrow {\scriptstyle \mathfrak{s}_{Groth,\pi}} & \\ & \mathbb{C} & \end{array}$$

- $\mathfrak{s}_{Groth,\pi}$ sends (ρ, W) to $Tr_W\left(\rho\left(t_\pi\right)\right)$ and $\mathfrak{s}_{Hecke,\pi}$ is given by the action of π on $V^{GL_2(\mathbb{Z}_p)}$, namely

$$\int_{GL_2(\mathbb{Q}_p)} f(g)\,\pi(g)\,v^0 dg = \mathfrak{s}_{Hecke,\pi}(f)\,v^0; \qquad \mathfrak{s}_{Hecke,\pi}(f) \in \mathbb{C}.$$

- This is a spectral decomposition.
- Apply $\ell \in \pi^\vee$ to that.
- The geometric basis: the characteristic functions centered on $GL_2\left(\mathbb{Z}_p\right)\lambda GL_2\left(\mathbb{Z}_p\right)$.
- The spectral basis: $A_\lambda = \mathfrak{s}\left(\rho_\lambda\right)$, where $\left(\rho_\lambda, V_\lambda\right)$ are irreducible.

Generating Function

Lemma(Existence of generating function)

There exists a unique $\Delta_s \in \mathcal{H}[[p^{-s}]]$ such that for any unramified representation π of $GL_2(\mathbb{Q}_p)$ and any $\ell \in \pi^\vee$ it holds that

$$\int_{GL_2(\mathbb{Q}_p)} \Delta_s(g)\,\ell\left(\pi(g)\varphi_p\right) = \mathcal{L}(s,\pi,\mathfrak{st})\,\ell\left(\varphi_p\right).$$

Namely, $\mathfrak{s}_{Hecke,\pi}(\Delta_s) = \mathcal{L}(s,\pi,\mathfrak{st})$

Proof.

If π is unramified then π is a subquotient of $\mathrm{Ind}_B^G \chi$, where $\chi\left(\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix}\right) = |t_1|_{\mathbb{Q}_p}^{\chi_1} |t_2|_{\mathbb{Q}_p}^{\chi_2}$ then $t_\pi = \begin{pmatrix} p^{\chi_1} & \\ & p^{\chi_2} \end{pmatrix}$.

Proof continued

$$\mathcal{L}(s,\pi,\mathfrak{st}) = (1-\chi_1 p^{-s})^{-1}(1-\chi_2 p^{-s})^{-1} = \left(\sum_{k=0}^{\infty} \chi_1^k p^{-ks}\right)\left(\sum_{k=0}^{\infty} \chi_2^k p^{-ks}\right)$$
$$= \sum_{k=0}^{\infty} p^{-ks}\left(\sum_{j=0}^{k} \chi_1^j \chi_2^{k-j}\right) = \sum_{k=0}^{\infty} p^{-ks} Tr\left(Sym^k(t_\pi)\right)$$
$$= \sum_{k=0}^{\infty} p^{-ks}\,\mathfrak{s}_{Groth,\pi}(\rho_{n\omega_1}) = \mathfrak{s}_{Hecke,\pi}\left(\sum_{k=0}^{\infty} p^{-ks} A_{k\omega_1}\right)$$

ω_1 is the first fundamental weight and $k\omega_1$ is the highest weight of Sym^k. Fundamental weights: $\omega_1\left(\begin{pmatrix} a & \\ & b\end{pmatrix}\right) = a, \omega_2\left(\begin{pmatrix} a & \\ & b\end{pmatrix}\right) = ab.$

Proof of the Local Statement

$$\begin{aligned}
\mathcal{L}(s,\pi,\mathrm{st}) &= \int_{GL_2(\mathbb{Q}_p)} \Delta_s(g)\,\ell(\pi(g)\varphi)\,dg \\
&= \int_{N(\mathbb{Q}_p)\backslash GL_2(\mathbb{Q}_p)} \int_{N(\mathbb{Q}_p)} \Delta_s(ng)\,\ell(\pi(ng)\varphi)\,dn\,dg \\
&= \int_{N(\mathbb{Q}_p)\backslash GL_2(\mathbb{Q}_p)} \left(\int_{N(\mathbb{Q}_p)} \Delta_s(g)\,\psi(n)\,dn \right) \ell(\pi(ng)\varphi)\;dg \\
&= \int_{N(\mathbb{Q}_p)\backslash GL_2(\mathbb{Q}_p)} \Delta_s^{\psi}(g)\,\ell(\pi(g)\varphi)\;dg = \int_{T(\mathbb{Q}_p)} \Delta_s^{\psi}(t)\,\ell(\pi(t)\varphi)\;dt
\end{aligned}$$

$$\mathcal{L}(s,\pi,\mathfrak{st}) = \int\limits_{T(\mathbb{Q}_p)} \Delta_s^{\psi}(t)\,\ell(\pi(t)\varphi)\ dt$$

$$= \sum_{k=0}^{\infty} p^{-ks} \int\limits_{T(\mathbb{Q}_p)} A_{k\omega_1}^{\psi}(t)\,\ell(\pi(t)\varphi)\ dt$$

Casselman-Shalika Formula [Frenkel, Gaitsgory, Kazhdan, Vilonen]

$$A_{\lambda}^{\psi} = \delta_B^{-\frac{1}{2}}\,\mathbb{1}_{(N,\psi)\lambda K}$$

$$= \sum_{k=0}^{\infty} p^{-ks} \int\limits_{T(\mathbb{Q}_p)} \delta_B^{-\frac{1}{2}}(t)\,\mathbb{1}_{(N,\psi)\left(p^{k} \quad {}_{1}\right)K}(t)\,\ell(\pi(t)\varphi)\ dt$$

$$\mathcal{L}(s,\pi,\mathfrak{st}) = \sum_{k=0}^{\infty} p^{-ks} \int\limits_{|a|=p^{-k}} p^{\frac{k}{2}} \ell\left(\pi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)\varphi\right) da$$

$$= \sum_{k=0}^{\infty} \int\limits_{|a|=p^{-k}} |a|^{s-\frac{1}{2}} \ell\left(\pi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)\varphi\right) da$$

$$= \int\limits_{\mathbb{Z}_p\setminus\{0\}} |a|^{s-\frac{1}{2}} \ell\left(\pi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)\varphi\right) da$$

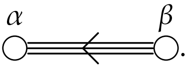
$$= \int\limits_{\mathbb{Q}_p^\times} |a|^{s-\frac{1}{2}} \ell\left(\pi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)\varphi\right) da$$

Since $\ell\left(\pi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)\varphi\right) = 0$ for $a \notin \mathbb{Z}_p$. □

Brake Time

The Exceptional Group of Type G_2

- G - the exceptional group of type G_2



- B - Borel subgroup with torus T and unipotent radical N
- $\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$
- P - Heisenberg parabolic subgroup corresponding to α, $P = M \cdot U$
- $M \cong GL_2$

Remark

$$G_2(F) = \mathrm{Aut}_F(\mathbb{O}(F))$$

Irreducible Cuspidal Representation and Partial $\mathcal{L}$-Functions

- $\pi = \bigotimes_{\nu \in \mathcal{P}} \pi_\nu$ - irreducible cuspidal representation of $G(\mathbb{A})$
- $\chi : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ - Hecke character
- $S \subset \mathcal{P}$ - Finite set such that π_ν and χ_ν are unramified for $\nu \notin S$
- For $\nu \notin S$ let $t_{\pi_\nu} \in {}^L G = G_2(\mathbb{C})$ be the Satake parameter of π_ν. $(t_{\pi_\nu}, \chi(\varpi_\nu))$ is the Satake parameter of $\pi_\nu \boxtimes \chi_\nu$ in ${}^L(G \times GL_1) = G_2(\mathbb{C}) \times \mathbb{C}^\times$.
- $\mathfrak{st}$ - Standard 7-dimensional representation of $G_2(\mathbb{C}) \times \mathbb{C}^\times$
- $\mathcal{L}(s, \pi_\nu, \chi_\nu, \mathfrak{st}) = \frac{1}{\det(\mathbb{1} - q_\nu^{-s} \chi_\nu(\varpi_\nu)\, \mathfrak{st}(t_{\pi_\nu}))}$
- $\mathcal{L}^S(s, \pi, \chi, \mathfrak{st}) = \prod_{\nu \notin S} \mathcal{L}(s, \pi_\nu, \chi_\nu, \mathfrak{st})$

Goal

1. Prove meromorphic continuation of $\mathcal{L}^S(s, \pi, \chi, \rho)$.
2. Study the poles of $\mathcal{L}^S(s, \pi, \chi, \rho)$.

Étale Cubic Algebras

An étale cubic algebra E over F is one of the following

- $E = F \times F \times F$ - Split case
- $E = F \times K$ - K is a field
- E is a field

Surprising Correspondence

$$\left\{\begin{matrix}\text{Iso. classes of}\\ \text{Quasi-split}\\ \text{forms of } D_4/F\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Iso. classes}\\ \text{of étale cubic}\\ \text{algebras over } F\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Non-degenerate}\\ M(F)\text{-orbits of}\\ \text{characters of } U(\mathbb{A})\end{matrix}\right\}$$

Notation: $H_E \leftrightarrow E \leftrightarrow \Psi_E$.

Fourier Coefficients

Definition

$$L_{\Psi_E}(\varphi)(g) = \int\limits_{U(F)\backslash U(\mathbb{A})} \varphi(ug)\overline{\Psi_E(u)}du$$

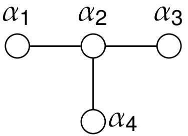
- We say π supports the Ψ_E-Fourier coefficient if $\exists \varphi \in \pi : \ L_{\Psi_E}(\varphi) \neq 0$.
- For any $g \in G(\mathbb{A})$ it holds that

$$L_{\Psi_E}(\cdot)(g) \in \mathrm{Hom}_{U(\mathbb{A})}\left(\pi, \mathbb{C}_{\Psi_E}\right)$$

Usually, $\dim \mathrm{Hom}_{U(\mathbb{A})}\left(\pi, \mathbb{C}_{\Psi_E}\right) > 1$

Quasi-Split Forms of D_4

- H_E - The quasi split form of type D_4 corresponding to E



$\circlearrowleft S_E = \mathrm{Aut}_F(E)$.

- $1 \to H_E \to \mathrm{Aut}(H_E) \to S_E \to 1$.
- $G = H_E^{S_E}$
- $B_E = T_E \cdot N_E$ with $B = B_E \cap G$
- $P_E = M_E \cdot U_E$ with $P = P_E \cap G$
- $M_E = \left\{ g \in \mathrm{Res}_{E/F}\, GL_2 \,\middle|\, \det(g) \in F^\times \right\}$

The Zeta Integral

- $I_{P_E}(\chi, s) = \operatorname{Ind}_{P_E}^{H_E} \chi \circ \det_{M_E} \otimes \left|\det_{M_E}\right|^{s+\frac{5}{2}}$
- For the normalized holomorphic section $f_s^* \in I_{P_E}(\chi, s)$ we define the normalized Eisenstein series

$$\mathcal{E}_E^*(\chi, s, f, h) = \sum_{\gamma \in P_E(F) \backslash H_E(F)} f_s^*(\gamma h) \qquad h \in H_E(\mathbb{A}).$$

- For $\varphi \in \pi$ let

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \int_{G(F) \backslash G(\mathbb{A}_F)} \varphi(g) \, \mathcal{E}_E^*(\chi, s, f, g) \, dg.$$

This defines a meromorphic function for any φ, f.

$f_s^*(1) = j_E(\chi, s)$

The Main Result

Theorem

Assume that π supports the Ψ_E-Fourier coefficient and assume that $S \subset \mathcal{P}$ is a finite subset such that for any $\nu \notin S$ all data is unramified. It holds that

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \mathcal{L}^S(s, \pi, \chi, \rho)\, d_S(\chi, s, \Psi_E, \varphi_S, f_S).$$

Moreover, for any s_0 data can be chosen so that $d_S(\chi, s, \Psi_E, \varphi_S, f_S)$ is analytic and non-vanishing in a neighborhood of s_0.

Corollary

$\mathcal{L}^S(s, \pi, \chi, \rho)$ admits a meromorphic continuation.

Unfolding

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \int\limits_{G(F)\backslash G(\mathbb{A})} \varphi(g)\, \mathcal{E}_E^*(\chi, s, f, g)\, dg =$$

$$= \sum_{\mu \in P_E(F)\backslash H_E(F)/G(F)} \int\limits_{\mathrm{Stab}_{G(F)}(\mu)\backslash G(\mathbb{A})} \varphi(g)\, f_s(\mu g)\, dg.$$

Problem

Describe $P_E(F)\backslash H_E(F)/G(F)$

Parametrization of $P_E\left(\overline{F}\right)\backslash H_E\left(\overline{F}\right)/G\left(\overline{F}\right)$

$$\begin{array}{ccc} \underset{m}{(GL_2\times GL_2\times GL_2)^0\left(\overline{F}\right)} & \xrightarrow{\quad\Upsilon\quad} & \underset{w_2 m}{P_E\left(\overline{F}\right)\backslash H_E\left(\overline{F}\right)}\,. \\ & \searrow \qquad \nearrow & \\ & (B_0\times B_0\times B_0)^0\left(\overline{F}\right)\backslash (GL_2\times GL_2\times GL_2)^0\left(\overline{F}\right) & \end{array}$$

- $M\left(\overline{F}\right)\cong GL_2\left(\overline{F}\right)$ action from the right
- Bruhat decomposition $\rightsquigarrow \left(B_0\left(\overline{F}\right)\backslash GL_2\left(\overline{F}\right)\right)\cong\mathbb{P}^1\left(\overline{F}\right)$

$$x\longleftrightarrow B_0\left(\overline{F}\right)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&x\\0&1\end{pmatrix},\quad \infty\longleftrightarrow B_0\left(\overline{F}\right).$$

- Möbius transformations $\mathbb{P}^1\left(\overline{F}\right)\times\mathbb{P}^1\left(\overline{F}\right)\times\mathbb{P}^1\left(\overline{F}\right)/GL_2\left(\overline{F}\right)$
  1. $\Upsilon(\infty,\infty,\infty)=w_2\sim 1$
  2. $\Upsilon(0,\infty,\infty)=w_2w_1,\quad \Upsilon(\infty,0,\infty)=w_2w_3,\quad \Upsilon(\infty,\infty,0)=w_2w_4$
  3. $w_2w_3x_{-\alpha_1}(1)=\Upsilon(1,0,\infty)$

  In fact this are representatives for $P_E\left(\overline{F}\right)\backslash H_E\left(\overline{F}\right)/G\left(\overline{F}\right)$

Galois Descent

Fix $\mu \in H_E(F) \subseteq H_E(\overline{F})$:

- $\{1\} \to \mathrm{Stab}_{G(\bar{F})}(\mu) \to G_2(\bar{F}) \to P_E(\overline{F})\mu G(\overline{F}) \to \{1\}$
-

$$\{1\} \longrightarrow \mathrm{Stab}_{G(\bar{F})}(\mu)^{\Gamma} \longrightarrow G_2(F) \longrightarrow P_E(F)\mu G(F) \longrightarrow \mathrm{H}^1\left(\Gamma_E, \mathrm{Stab}_{G(\bar{F})}(\mu)\right) \longrightarrow \mathrm{H}^1\left(\Gamma_E, G_2(\bar{F})\right) \longrightarrow \mathrm{H}^1\left(\Gamma_E, P_E(\overline{F})\mu G(\overline{F})\right)$$

- $P_E(F)\mu G(F)/G(F) \longleftrightarrow \mathrm{Ker}\left[\mathrm{H}^1\left(\Gamma_E, \mathrm{Stab}_{G(F)}(\mu)\right) \to \mathrm{H}^1(\Gamma_E, G(F))\right]$
- G_2 is a split reductive group $\Rightarrow \mathrm{H}^1(\Gamma_E, G(F)) = \{1\}$
- R unipotent $\Rightarrow \mathrm{H}^1(\Gamma_E, R(F)) = \{1\}$
- Hilbert 90': $\Rightarrow \mathrm{H}^1(\Gamma_E, GL_n(F)) = \{1\}$

$$\mathrm{H}^1\left(\Gamma_E, \mathrm{Stab}_{G(F)}(\mu)\right) = \{1\}$$

- $P_E(F)\mu G(F)/G(F) \longleftrightarrow \{1\}$
- If $\mu \in H_E(F)$ then $P_E(F)\mu G(F)/G(F) = \{\mu\}$

Factorization of $\mathcal{Z}_E(\chi, s, \varphi, f)$

We fix a representative $\mu \in H_E(F)$ for the open orbit

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \int_{\mathrm{Stab}_{G(F)}(\mu)\backslash G(\mathbb{A})} \varphi(g) f_s(\mu g)\, dg = \int_{U(\mathbb{A})\backslash G(\mathbb{A})} L_{\Psi_E}(\varphi)(g) F^*(\Psi_E, \chi, g, s)\, dg$$

Obstacle

As (usually) $\dim \mathrm{Hom}_{U(\mathbb{A})}(\pi, \Psi_E) > 1$, the integral is not factorizable in the usual sense.

(Refined) Unramified Calculation

For $\nu \notin S$ let v_0 be a spherical vector in π_ν. There exists $s_0 \in \mathbb{R}$ such that for any $\Re\mathfrak{e}(s) > s_0$ and **any** $\Lambda \in \mathrm{Hom}_{U(F_\nu)}\left(\pi_\nu, \mathbb{C}_{\Psi_{E,\nu}}\right)$ it holds that

$$\int_{U(F_\nu)\backslash G(F_\nu)} F_\nu^*(\Psi_{E,\nu}, \chi_\nu, g, s) \Lambda(\pi_\nu(g) v_0)\, dg = \mathcal{L}\left(s + \frac{1}{2}, \pi_\nu, \chi_\nu, \mathfrak{st}\right) \Lambda(v_0)$$

The Generating Functions

We fix $\nu \notin S$ and drop ν from all notations for this discussion.
$\mathcal{H}$ - spherical Hecke algebra of $G(F)$ with respect to $G(\mathcal{O})$.

Existence of Generating Functions

$$\exists \Delta_{\chi,s} \in \mathcal{H}[[q^{-s}]] : \quad \int_{G(F)} \Delta_{\chi,s}(g) \Lambda(\pi(g) v_0)\, dg = \mathcal{L}(s, \pi, \chi, \mathfrak{st}) \Lambda(v_0) \quad \forall \Lambda \in \pi^*,\ s \gg 0$$

Goal

$$\int_{U(F)\backslash G(F)} F^*(\Psi_E, \chi, g, s) \Lambda(\pi(g) v_0)\, dg = \int_{U(F)\backslash G(F)} \Delta^{\Psi_E}_{\chi, s+\frac{1}{2}}(g) \Lambda(\pi(g) v_0)\, dg$$

Better yet, prove $F^*(\Psi_E, \chi, g, s) = \Delta^{\Psi_E}_{\chi, s+\frac{1}{2}}(g)$.

Obstacle

$\Delta_{\chi,s}(g)$ is complicated.

Approximations to Generating Functions

$$D_s \in \mathcal{H}\left[\left[q^{-s}\right]\right] \quad D_s\left(ktk'\right) = \left|\omega_1(t)\right|^{s+\frac{7}{2}} \qquad \forall t \in T^+,\ k, k' \in K$$

Theorem

$$\exists P_s \in \mathcal{H}\left[q^{-s}\right] : \quad D_s = \Delta_{s+\frac{1}{2}} * P_s \quad \forall\, \Re\mathfrak{e}\, s \gg 0.$$

More precisely $P_s = \dfrac{P_0\left(q^{-s-\frac{1}{2}}\right)A_0 - P_1\left(q^{-s-\frac{1}{2}}\right)A_1}{\zeta_F\left(s+\frac{3}{2}\right)\zeta_F\left(s+\frac{7}{2}\right)\zeta_F\left(s+\frac{1}{2}\right)}$.
Furthermore, for $\Re\mathfrak{e}\,(s) \gg 0$ the operator $*P_s$ is injective.

Proof (of the second part)

$\mathcal{H}\left[q^{-s}\right] \subset L^1\left(G\left(F\right)\right)$ and hence $\mathcal{H}\left[q^{-s}\right] \hookrightarrow \mathcal{B}\left(L^2\left(G\left(F\right)\right)\right)$.
As $\dfrac{P_1\left(q^{-s-\frac{1}{2}}\right)}{P_0\left(q^{-s-\frac{1}{2}}\right)} \underset{\Re\mathfrak{e}(s)\to\infty}{\longrightarrow} 0$ this proves the claim.

Theorem

1. $D_s^{\Psi_E} \equiv F^* \left(\Psi_E, \chi, \cdot, s\right) * P_s$
2. $D_s^{\Psi_E} = \Delta_{s+\frac{1}{2}}^{\Psi_E} * P_s$

Corollary

$$\int\limits_{U(F)\backslash G(F)} F^* \left(\Psi_E, \chi, g, s\right) \Lambda \left(\pi\left(g\right) v_0\right) dg = \mathcal{L}\left(s + \frac{1}{2}, \pi, \chi, \mathfrak{st}\right) \Lambda \left(v_0\right)$$

Poles of $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$

$$\int_{G(F)\backslash G(\mathbb{A})} \varphi(g)\,\mathcal{E}_E^*(\chi,s,f,g)\,dg = \mathcal{L}^S(s,\pi,\chi,\mathfrak{st})\,d_S(\chi,s,\Psi_E,\varphi_S,f_S)$$

$$\Rightarrow \mathrm{ord}_{s=s_0}\,\mathcal{L}^S(s,\pi,\chi,\mathfrak{st}) \le \mathrm{ord}_{s=s_0}\,\mathcal{E}_E^*(\chi,s,f,g)\,.$$

The Poles of $\mathcal{E}_E(\chi,f_s,s,g)$

	$s=\frac{1}{2}$	$s=\frac{3}{2}$		$s=\frac{5}{2}$
	χ quad.	$\chi=1$	$\chi=\chi_E$	$\chi=1$
$E=F\times F\times F$	1	2	-	1
$E=F\times K$	1	1	1	1
E Galois field extension	1	0	1	1
E non-Galois	1	0	-	1

Conjecture (Following Arthur's conjectures)

For any (χ,s_0), with $s_0=\frac{1}{2},\frac{3}{2}$, such that $\mathcal{E}_E$ admits a pole of order n_0 there exist a a cuspidal representation π of $G(\mathbb{A})$ supporting the Ψ_E Fourier coefficient such that $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ admits a pole of order n_0 at s_0.

Shadows of Eisenstein Series

CAP representations

Let $P = M \cdot N \subset G$ be a parabolic subgroup, σ be a cuspidal unitary representation of the Levi part M and χ be a character of M. A cuspidal representation π of $G(\mathbb{A})$ is called **CAP** with respect to (P, σ, χ) if π is nearly equivalent to a subquotient of $\mathrm{Ind}_P^G \sigma \otimes \chi$.

Theorem

Let E be a Galois cubic étale algebra over F. Let n_E be 2 if $E = F \times F \times F$ and 1 otherwise. The following are equivalent

1. $\mathcal{L}^S(s, \pi, \chi_E, \mathfrak{st})$ admits a pole at $s = 2$ of order n_E.
2. $\Theta_{S_E}(\pi) \neq 0$. In particular π is nearly equivalent to $\Theta_E(1)$, where 1 here is the automorphic trivial representation of $S_E(\mathbb{A})$.
3. π is a **CAP** representation with respect to B supporting the Ψ_E-Fourier coefficient.

Thank You!