L-Groups

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$$GL_n \longleftrightarrow GL_n$$
$$SL_n \longleftrightarrow PGL_n$$
$$SO_{2n} \longleftrightarrow SO_{2n}$$
$$Sp_{2n} \longleftrightarrow SO_{2n+1}$$

F - local non-archimedean field
$\mathcal{O} = \mathcal{O}_F$ - ring of integers

G - connected split reductive group $/F$
B - Borel subgroup of G
T - maximal split torus
W - Weyl group

Motivation

The **spherical Hecke algebra** is defined as

$$\mathcal{H} = C_c^\infty\left(K\backslash G\left(F\right)/K\right)$$

together with the multiplication

$$f_1 * f_2\left(g\right) = \int\limits_{G(F)} f_1\left(gh^{-1}\right) f_2\left(h\right) dh$$

Fact

This is a unital algebra with $1_{\mathcal{H}} = e_K$.

Theorem

$\mathcal{H}$ is a commutative algebra.

We then have

Theorem (Satake Isomporphism - Version I)

$$\mathcal{H} \cong \mathbb{C}\left[X_*\left(T\right)\right]^W$$

Let $\mathbb{T} = X^*\left(T\right) \otimes_{\mathbb{Z}} \mathbb{C}$, then $X^*\left(\mathbb{T}\right) \cong X_*\left(T\right)$ and $X_*\left(\mathbb{T}\right) \cong X^*\left(T\right)$.
If $G = T$, the theorem yields

$$\mathcal{H} \cong \mathrm{Rep}\left(\mathbb{T}\right).$$

Goal

There exists a complex reductive group ${}^L G$ such that

$$\mathcal{H} \cong \mathrm{Rep}\left({}^L G\right)$$

and such that $\mathbb{T}$ is a maximal torus of ${}^L G$.

Root Datum

A **root datum** is a quadruple $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$, where

1. X, $X^\vee$ - free $\mathbb{Z}$-modules of finite rank
2. perfect pairing $X \times X^\vee \to \mathbb{Z}$ denoted by $\langle , \rangle$,
3. $\Phi \subset X$, $\Phi^\vee \subset \Phi^\vee$ - finite subsets and
4. there is a bijection

$$\Phi \to \Phi^\vee$$
$$\alpha \mapsto \alpha^\vee,$$

satisfying $\forall \alpha \in \Phi$

RD1 $\langle \alpha, \alpha^\vee \rangle = 2$

RD2 $s_\alpha \Phi = \Phi, \ s_{\alpha^\vee} \Phi^\vee = \Phi^\vee$

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$$
$$s_{\alpha^\vee}(y) = y - \langle \alpha, y \rangle \alpha^\vee$$

Root Datum Attached to a Reductive Group

$$G \rightsquigarrow \Psi = \Psi(G) = \Psi(G, T) = (X, \Phi, X^\vee, \Phi^\vee)$$

$$X = X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$$
$$X^\vee = X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T).$$

$$\chi \in X, \phi \in X^\vee \ \chi \circ \phi \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$$

We set

$$\langle \chi, \phi \rangle = \chi \circ \phi \in \mathbb{Z}$$
$$\chi(\phi(t)) = t^{\langle \chi, \phi \rangle}$$

$\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{h} \oplus \bigoplus_{\alpha \in X} \mathfrak{g}_\alpha$, where $\mathfrak{h} = \mathrm{Lie}(T)$ and T acts on $\mathfrak{g}_\alpha$ by α.

$$\Phi = \{\alpha \mid \mathfrak{g}_\alpha \neq \{0\}\}$$

$\alpha \rightsquigarrow s_\alpha$

$$\exists! \alpha^\vee \in X^\vee : \quad \forall x \in X : \ s_\alpha(x) = x - \left\langle x, \alpha^\vee \right\rangle \alpha$$

$$\Phi^\vee = \left\{\alpha^\vee \,\middle|\, \alpha \in \Phi\right\}$$

Existence and Uniqueness Theorem

$\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ is **reduced** if

$$\forall \alpha \in \Phi, c \in \mathbb{Q} \quad c\alpha \in \Phi \Rightarrow c = \pm 1.$$

Theorem (Springer theorems 9.6.2 and 10.1.1)

There is a bijection

$$\{\textit{Split connected reductive groups over } k\} \leftrightarrow \{\textit{Reduced root datum}\}$$

taking a group G to $\Psi(G)$.

Dual Root Datum and the L-group

From $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ we form the **dual root datum**

$$\Psi^\vee = \left(X^\vee, \Phi^\vee, X, \Phi\right)$$

$\rightsquigarrow$ There is a unique $G^\vee$ defined over $\mathbb{C}$ - **the dual group**

$$\left(\Psi^\vee\right)^\vee = \Psi$$

Example I - $G = GL_2$

$$T = \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \,\middle|\, t_1, t_2 \in k^\times \right\}.$$

$$\chi_1 \left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \right) = t_1, \quad \chi_2 \left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \right) = t_2$$

$$\phi_1 (t_1) = \begin{pmatrix} t_1 & \\ & 1 \end{pmatrix}, \quad \phi_2 (t_2) = \begin{pmatrix} 1 & \\ & t_2 \end{pmatrix}.$$

$$X = X^* (T) = \left\{ \chi_1^n \chi_2^m \,\middle|\, m, n \in \mathbb{Z} \right\}$$

$$X^\vee = X_* (T) = \left\{ \phi_1^n \phi_2^m \,\middle|\, m, n \in \mathbb{Z} \right\}$$

$$\left\langle \chi_1, \phi_1 \right\rangle = 1, \ \left\langle \chi_1, \phi_2 \right\rangle = 0, \ \left\langle \chi_2, \phi_1 \right\rangle = 0, \ \left\langle \chi_2, \phi_2 \right\rangle = 1$$

$$\Phi = \{\alpha, -\alpha\}, \quad \alpha = \chi_1 \chi_2^{-1}, \quad \Phi^\vee = \left\{ \alpha^\vee, -\alpha^\vee \right\}, \quad \alpha^\vee = \phi_1 \phi_2^{-1}$$

This root datum is obviously self dual and hence $GL_2^\vee = GL_2$.

Maps of Root Data

Given $\Psi_1 = \left(X_1, \Phi_1, X_1^\vee, \Phi_1^\vee\right)$ and $\Psi_2 = \left(X_2, \Phi_2, X_2^\vee, \Phi_2^\vee\right)$ a map $\Psi_1 \to \Psi_2$ is

$$f : X_2 \to X_1$$

such that

1. $f|_{\Phi_2}$ is a bijection onto Φ_1
2. $\forall \alpha \in \Phi_2 \; f(\alpha)^\vee = \alpha^\vee$

If f is a bijection, we say that Ψ_1 and Ψ_2 are isomorphic. This is equivalent to isomorphism in the level of groups.

Example II - $G = SL_2$

$$T = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \middle| \, t \in k^\times \right\}$$

$$\begin{aligned} X &= X^*(T) = \left\{ \chi_1^n \,\middle|\, n \in \mathbb{Z} \right\} \\ X^\vee(T) &= \left\{ \phi^n \,\middle|\, n \in \mathbb{Z} \right\}, \; \phi = \phi_1 \phi_2^{-1} \\ \left\langle \chi_1, \phi \right\rangle &= 1 \\ \Phi &= \{\alpha, -\alpha\}, \; \alpha = \chi_1^2 \\ \Phi^\vee &= \left\{ \alpha^\vee, -\alpha^\vee \right\}, \; \alpha^\vee = \phi. \end{aligned}$$

This root datum is obviously not self dual since

$$\mathbb{Z} \cdot \Phi \subsetneq X, \; \mathbb{Z} \cdot \Phi^\vee = X^\vee.$$

How can we find $SL_2^\vee$?

1. We can use the proof of the existence theorem (which is constructive) and construct $SL_2^\vee$ as a Chevalley group.
2. If one knows a list of algebraic groups and their root datum he can try and recover the dual group from the list.

We pursue the second option.

Root Datum of PGL_2

$$R = \left\{ \begin{pmatrix} t & \\ & 1 \end{pmatrix} \,\middle|\, t \in k^\times \right\}$$

$$Y = X^*(R) = \left\{ \chi_1^n \,\middle|\, n \in \mathbb{Z} \right\}$$
$$Y^\vee = X_*(R) = \left\{ \phi_1^n \,\middle|\, n \in \mathbb{Z} \right\}$$
$$\Xi = \{\beta, -\beta\}, \quad \beta = \chi_1$$
$$\Xi^\vee = \left\{\beta^\vee, -\beta^\vee\right\}, \quad \beta^\vee = \phi_1^2$$

Let

$$f : Y \to X^\vee$$
$$\chi_1 \mapsto \phi = \phi_1 \phi_2^{-1}$$

Then

$$f^\vee : X \to Y^\vee$$
$$\chi_1 \mapsto \phi_1$$

and in particular

$$f(\beta) = \alpha^\vee,\ f(\beta)^\vee = \alpha.$$

Hence

$$\left(X^\vee, \Phi^\vee, X, \Phi\right) \cong \left(Y, \Xi, Y^\vee, \Xi^\vee\right),$$

and

$$SL_2^\vee = PGL_2.$$

L-Groups of Classical Groups

$$GL_n \longleftrightarrow GL_n \qquad (A_n)$$

$$SL_n \longleftrightarrow PGL_n \qquad (A_n)$$

$$Sp_{2n} \longleftrightarrow SO_{2n+1} \qquad (B_n \leftrightarrow C_n)$$

$$SO_{2n} \longleftrightarrow SO_{2n} \qquad (D_n)$$

$$Spin_{2n+1} \longleftrightarrow Sp_{2n}/Z \qquad (B_n \leftrightarrow C_n)$$

$$Spin_{2n} \longleftrightarrow SO_{2n}/Z \qquad (D_n)$$

$$\text{Adjoint} \longleftrightarrow \text{Simply connected}$$

G is called **adjoint** if $X^*(T) = \mathbb{Z}[\Phi]$, example: PGL_2.
G is called **simply connected** if $X_*(T) = \mathbb{Z}[\Phi^\vee]$, example: SL_2.

Back to Satake Isomorphism

Theorem (Satake Isomorphism - Version II)

For G split

$$\mathcal{H} \cong \mathrm{Rep}\left(G^{\vee}(\mathbb{C})\right)$$

We can dualize this

Theorem (Satake Isomorphism - Version III)

There is bijection

$$\{\chi : \mathcal{H} \to \mathbb{C}\} \leftrightarrow T^{\vee}(\mathbb{C})/W \leftrightarrow \left\{\textit{semisimple conjugacy classes in } G^{\vee}(\mathbb{C})\right\}$$

The Non-split Case

We would like to generalize the Satake isomorphism to the case of a non-split case.
F - local non-archimedean field
$\overline{F}$ its algebraic closure
$\Gamma = \mathrm{Gal}\left(\overline{F}/F\right)$
G - reductive group defined over F

Facts

- $G_{\overline{F}}$ is split.
- $\exists \Gamma \circlearrowright G_{\overline{F}} : \; G = G_{\overline{F}}^{\Gamma}$

$$G_{\overline{F}} \rightsquigarrow \Psi\left(G_{\overline{F}}\right),\; G_{\overline{F}}^{\vee}$$

$$\Gamma \circlearrowright G_{\overline{F}} \rightsquigarrow \Gamma \circlearrowright \Psi\left(G_{\overline{F}}\right) \rightsquigarrow \Gamma \circlearrowright \Psi\left(G_{\overline{F}}\right)^{\vee} \rightsquigarrow \Gamma \circlearrowright G_{\overline{F}}^{\vee}$$

$$^{L}G = G_{\overline{F}}^{\vee} \rtimes \mathrm{Gal}\left(\overline{F}/F\right)$$

Theorem (Satake Isomorphism - Version III in General)

There is bijection

$$\{\chi : \mathcal{H} \to \mathbb{C}\} \leftrightarrow T^{\vee}(\mathbb{C})/W \leftrightarrow \left\{\textit{semisimple conjugacy classes in } {}^{L}G(\mathbb{C})\right\}$$

Example - $U(3)$

E/F - quadratic extension
$\Gamma = \mathrm{Gal}(E/F) = \{1, \sigma\}$
On E^3 we define a sesqui-linear form by

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1 \sigma(y_1) + x_2 \sigma(y_2) + x_3 \sigma(y_3)$$

$$U(n) = \left\{ g \in \mathrm{GL}_n(E) \,\middle|\, \forall u, v \in E^n \ \langle u, v \rangle = \langle g \cdot u, g \cdot v \rangle \right\},$$

We now compute ${}^L U(3)$.

$$U(3)_E \cong \mathrm{GL}_3$$

$$\langle u, v\rangle = \langle g \cdot u, g \cdot v\rangle = \left\langle u, \left(\sigma\left({}^{T}g\right)g\right) \cdot v\right\rangle$$

$$U(3) = GL_3^{\mathrm{Gal}(E/F)}, \quad \sigma \cdot g = \sigma\left({}^{T}g^{-1}\right)$$

The corresponding action of $\mathrm{Gal}(E/F)$ on $GL_3^{\vee} = GL_3$ is given by

$$\sigma \cdot g = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} {}^{T}g^{-1} \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}.$$

We conclude by setting

$${}^{L}U(3) = GL_3(\mathbb{C}) \rtimes \Gamma$$

Thank You!