Jacquet Modules and Irreducibility of Parabolic Induction

Avner Segal
(Joint with Hezi Halawi)

Department of Mathematics
Bar-Ilan University
segalavner@gmail.com

Induction

Notations

- $\mathbb{Q}_p$ - the field of p-adic numbers.
- G - a split, reductive, p-adic group (e.g. $GL_n\left(\mathbb{Q}_p\right)$, $SL_n\left(\mathbb{Q}_p\right)$, $Sp_{2n}\left(\mathbb{Q}_p\right)$, split exceptional groups...)

Notations

- $\mathbb{Q}_p$ - the field of p-adic numbers.
- G - a split, reductive, p-adic group (e.g. $GL_n(\mathbb{Q}_p)$, $SL_n(\mathbb{Q}_p)$, $Sp_{2n}(\mathbb{Q}_p)$, split exceptional groups...)
- Fix a Borel subgroup $B = TN$ of G, where T is a maximal split torus and N is its unipotent radical .

Notations

- $\mathbb{Q}_p$ - the field of p-adic numbers.
- G - a split, reductive, p-adic group (e.g. $GL_n\left(\mathbb{Q}_p\right)$, $SL_n\left(\mathbb{Q}_p\right)$, $Sp_{2n}\left(\mathbb{Q}_p\right)$, split exceptional groups...)
- Fix a Borel subgroup $B = TN$ of G, where T is a maximal split torus and N is its unipotent radical .
- We write $P = MU$ for a standard parabolic subgroup of G with Levi subgroup M and unipotent radical U

Notations

- $\mathbb{Q}_p$ - the field of p-adic numbers.
- G - a split, reductive, p-adic group (e.g. $GL_n(\mathbb{Q}_p)$, $SL_n(\mathbb{Q}_p)$, $Sp_{2n}(\mathbb{Q}_p)$, split exceptional groups...)
- Fix a Borel subgroup $B = TN$ of G, where T is a maximal split torus and N is its unipotent radical .
- We write $P = MU$ for a standard parabolic subgroup of G with Levi subgroup M and unipotent radical U, namely, $B \subseteq P \subseteq G$, $T \subseteq M \subseteq G$ and $U \subseteq N$.

Notations

- $\mathbb{Q}_p$ - the field of p-adic numbers.
- G - a split, reductive, p-adic group (e.g. $GL_n\left(\mathbb{Q}_p\right)$, $SL_n\left(\mathbb{Q}_p\right)$, $Sp_{2n}\left(\mathbb{Q}_p\right)$, split exceptional groups...)
- Fix a Borel subgroup $B = TN$ of G, where T is a maximal split torus and N is its unipotent radical .
- We write $P = MU$ for a standard parabolic subgroup of G with Levi subgroup M and unipotent radical U, namely, $B \subseteq P \subseteq G$, $T \subseteq M \subseteq G$ and $U \subseteq N$.
- Standard parabolic subgroups are in one-to-one correspondence with Dynkin sub-diagrams.

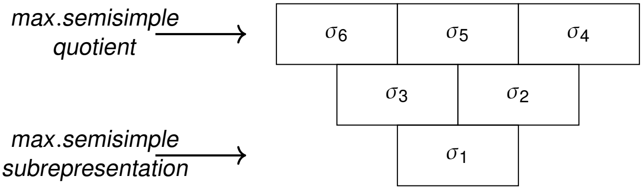
Notations

- $\mathbb{Q}_p$ - the field of p-adic numbers.
- G - a split, reductive, p-adic group (e.g. $GL_n\left(\mathbb{Q}_p\right)$, $SL_n\left(\mathbb{Q}_p\right)$, $Sp_{2n}\left(\mathbb{Q}_p\right)$, split exceptional groups...)
- Fix a Borel subgroup $B = TN$ of G, where T is a maximal split torus and N is its unipotent radical .
- We write $P = MU$ for a standard parabolic subgroup of G with Levi subgroup M and unipotent radical U, namely, $B \subseteq P \subseteq G$, $T \subseteq M \subseteq G$ and $U \subseteq N$.
- Standard parabolic subgroups are in one-to-one correspondence with Dynkin sub-diagrams.
- Let $Rep\left(G\right)$ denote the category of algebraic representations of G and let $\mathfrak{R}\left(G\right)$ denote the associated Grothendieck ring.

Irreducibles are the Building Blocks of *Rep* (G)

A **subrepresentation** of π is a G-invariant subspace $\pi_1 \subseteq \pi$. A **subquotient** of π is any representation of the form $\pi_1 \backslash \pi_2$ for some subrepresentations $\pi_1 \subset \pi_2 \subset \pi$.

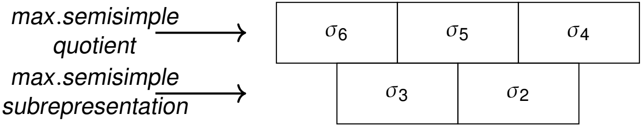
Example: π is indecomposable (not semi-simple) of length 6



Irreducibles are the Building Blocks of *Rep* (G)

A **subrepresentation** of π is a G-invariant subspace $\pi_1 \subseteq \pi$. A **subquotient** of π is any representation of the form $\pi_1 \backslash \pi_2$ for some subrepresentations $\pi_1 \subset \pi_2 \subset \pi$.

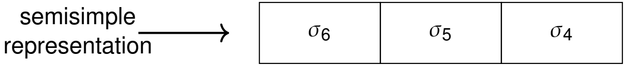
Example: $\pi' = \pi / \sigma_1$ is indecomposable of length 5



Irreducibles are the Building Blocks of *Rep* (G)

A **subrepresentation** of π is a G-invariant subspace $\pi_1 \subseteq \pi$. A **subquotient** of π is any representation of the form $\pi_1 \backslash \pi_2$ for some subrepresentations $\pi_1 \subset \pi_2 \subset \pi$.

Example: $\pi'' = \pi' / (\sigma_2 \oplus \sigma_3) = \pi / (\sigma_1 + \sigma_2 + \sigma_3)$ is semi-simple of length 3

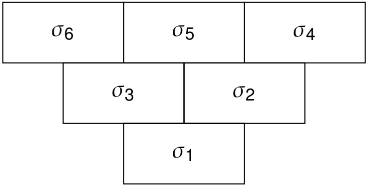


Irreducibles are the Building Blocks of $Rep(G)$

A **subrepresentation** of π is a G-invariant subspace $\pi_1 \subseteq \pi$. A **subquotient** of π is any representation of the form $\pi_1 \backslash \pi_2$ for some subrepresentations $\pi_1 \subset \pi_2 \subset \pi$.

Example: π is indecomposable (not semi-simple) of length 6

$\pi =$



We have the functor of semisimplification $s.s. : Rep(G) \to \mathfrak{R}(G)$:

$s.s.(\pi) =$

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6

Parabolic Induction and Jacquet Functors

Parabolic Induction

- For a parabolic subgroup $P = MU$ and a smooth representation (Ω, V_Ω) of M, let $i_M^G\Omega$ be the G-representation given by right-action

$$i_M^G\Omega = \mathrm{Ind}_P^G(\Omega \boxtimes \mathbf{1}) = \left\{ \begin{array}{c} \text{smooth functions } f : G \to V_\Omega \text{ satisfying} \\ f(mug) = \delta_P(m)^{1/2}\,\Omega(m)\,f(g) \\ \forall m \in M,\ u \in U,\ g \in G \end{array} \right\}.$$

Parabolic Induction

- For a parabolic subgroup $P = MU$ and a smooth representation (Ω, V_Ω) of M, let $i_M^G\Omega$ be the G-representation given by right-action

$$i_M^G\Omega = \mathrm{Ind}_P^G(\Omega \boxtimes \mathbf{1}) = \left\{ \begin{array}{c} \text{smooth functions } f : G \to V_\Omega \text{ satisfying} \\ f(mug) = \delta_P(m)^{1/2}\,\Omega(m)\,f(g) \\ \forall m \in M,\ u \in U,\ g \in G \end{array} \right\}.$$

- Parabolic induction is an exact functor $i_M^G : Rep(M) \to Rep(G)$ sending unitary representations to unitary representations.

Parabolic Induction

- For a parabolic subgroup $P = MU$ and a smooth representation (Ω, V_Ω) of M, let $i_M^G\Omega$ be the G-representation given by right-action

$$i_M^G\Omega = \mathrm{Ind}_P^G(\Omega \boxtimes \mathbf{1}) = \left\{ \begin{array}{c} \text{smooth functions } f : G \to V_\Omega \text{ satisfying} \\ f(mug) = \delta_P(m)^{1/2}\,\Omega(m)\,f(g) \\ \forall m \in M,\ u \in U,\ g \in G \end{array} \right\}.$$

- Parabolic induction is an exact functor $i_M^G : Rep(M) \to Rep(G)$ sending unitary representations to unitary representations.
- If $L \subset M$ is another Levi subgroup of G, then for any representation Ω of L we can induce in stages:

$$i_L^G\Omega \cong i_M^G \circ i_L^M\Omega.$$

Questions

- Is the induced representation $i_M^G \Omega$ irreducible (i.e. simple) or not?

Questions

- Is the induced representation $i_M^G\Omega$ irreducible (i.e. simple) or not?
- If $i_M^G\Omega$ is reducible, what are its composition factors?

Questions

- Is the induced representation $i_M^G \Omega$ irreducible (i.e. simple) or not?
- If $i_M^G \Omega$ is reducible, what are its composition factors?

Why are these irreducible factors interesting?

- Every non-cuspidal irreducible representation of G is constructed in this way.

Questions

- Is the induced representation $i_M^G \Omega$ irreducible (i.e. simple) or not?
- If $i_M^G \Omega$ is reducible, what are its composition factors?

Why are these irreducible factors interesting?

- Every non-cuspidal irreducible representation of G is constructed in this way.
- Needed for constructing the unitary dual of G.

Questions

- Is the induced representation $i_M^G \Omega$ irreducible (i.e. simple) or not?
- If $i_M^G \Omega$ is reducible, what are its composition factors?

Why are these irreducible factors interesting?

- Every non-cuspidal irreducible representation of G is constructed in this way.
- Needed for constructing the unitary dual of G. In particular, this helps in decomposing the structure of $L^2(X)$, where X is a measurable G-set.

Questions

- Is the induced representation $i_M^G \Omega$ irreducible (i.e. simple) or not?
- If $i_M^G \Omega$ is reducible, what are its composition factors?

Why are these irreducible factors interesting?

- Every non-cuspidal irreducible representation of G is constructed in this way.
- Needed for constructing the unitary dual of G. In particular, this helps in decomposing the structure of $L^2(X)$, where X is a measurable G-set.
- Plays a role in the construction of the residual automorphic spectrum of $G(\mathbb{A})$.

Questions

- Is the induced representation $i_M^G \Omega$ irreducible (i.e. simple) or not?
- If $i_M^G \Omega$ is reducible, what are its composition factors?

Why are these irreducible factors interesting?

- Every non-cuspidal irreducible representation of G is constructed in this way.
- Needed for constructing the unitary dual of G. In particular, this helps in decomposing the structure of $L^2(X)$, where X is a measurable G-set.
- Plays a role in the construction of the residual automorphic spectrum of $G(\mathbb{A})$.
- Important for studying automorphic/local $\mathcal{L}$-functions and functoriality.

Jacquet Functors

Theorem (Jacquet Functors)

- $\exists! \; r_M^G : Rep\,(G) \to Rep\,(M) : \quad \mathrm{Hom}_G\left(\pi, i_M^G \sigma\right) \cong \mathrm{Hom}_M\left(r_M^G \pi, \sigma\right).$

Jacquet Functors

Theorem (Jacquet Functors)

- $\exists!\ r_M^G : Rep\,(G) \to Rep\,(M) : \quad \mathrm{Hom}_G\left(\pi, i_M^G\sigma\right) \cong \mathrm{Hom}_M\left(r_M^G\pi, \sigma\right).$
- r_M^G is exact and satisfies $r_L^G = r_M^G \circ r_L^M$.

Jacquet Functors

Theorem (Jacquet Functors)

- $\exists!\ r_M^G : Rep\,(G) \to Rep\,(M) : \quad \mathrm{Hom}_G\left(\pi, i_M^G\sigma\right) \cong \mathrm{Hom}_M\left(r_M^G\pi, \sigma\right).$
- r_M^G is exact and satisfies $r_L^G = r_M^G \circ r_L^M$.

Cuspidal Representation and Cuspidal Support

- If π is irreducible and there exists a non-zero map $r_M^G\pi \to \sigma$, then there exists an embedding $\pi \hookrightarrow i_M^G\sigma$.

Jacquet Functors

Theorem (Jacquet Functors)

- $\exists!\ r_M^G : Rep\,(G) \to Rep\,(M) : \quad \mathrm{Hom}_G\left(\pi, i_M^G\sigma\right) \cong \mathrm{Hom}_M\left(r_M^G\pi, \sigma\right).$
- r_M^G is exact and satisfies $r_L^G = r_M^G \circ r_L^M$.

Cuspidal Representation and Cuspidal Support

- If π is irreducible and there exists a non-zero map $r_M^G\pi \to \sigma$, then there exists an embedding $\pi \hookrightarrow i_M^G\sigma$.
- An admissible representation π of G is said to be **cuspidal** if $r_M^G\pi = 0$ for all proper Levi subgroups of G.

Jacquet Functors

Theorem (Jacquet Functors)

- $\exists!\ r_M^G : Rep\,(G) \to Rep\,(M) : \quad \mathrm{Hom}_G\left(\pi, i_M^G \sigma\right) \cong \mathrm{Hom}_M\left(r_M^G \pi, \sigma\right)$.
- r_M^G is exact and satisfies $r_L^G = r_M^G \circ r_L^M$.

Cuspidal Representation and Cuspidal Support

- If π is irreducible and there exists a non-zero map $r_M^G \pi \to \sigma$, then there exists an embedding $\pi \hookrightarrow i_M^G \sigma$.
- An admissible representation π of G is said to be **cuspidal** if $r_M^G \pi = 0$ for all proper Levi subgroups of G. In this case, if $\pi \leq i_M^G \Omega$, then $M = G$ and $\sigma = \pi$.

Jacquet Functors

Theorem (Jacquet Functors)

- $\exists!\ r_M^G : Rep\,(G) \to Rep\,(M) : \quad \mathrm{Hom}_G\left(\pi, i_M^G\sigma\right) \cong \mathrm{Hom}_M\left(r_M^G\pi, \sigma\right)$.
- r_M^G is exact and satisfies $r_L^G = r_M^G \circ r_L^M$.

Cuspidal Representation and Cuspidal Support

- If π is irreducible and there exists a non-zero map $r_M^G\pi \to \sigma$, then there exists an embedding $\pi \hookrightarrow i_M^G\sigma$.
- An admissible representation π of G is said to be **cuspidal** if $r_M^G\pi = 0$ for all proper Levi subgroups of G. In this case, if $\pi \leq i_M^G\Omega$, then $M = G$ and $\sigma = \pi$.
- For any irreducible π, there exists a Levi M and cuspidal representation σ of M such that $\pi \hookrightarrow i_M^G\sigma$.

Composition of Adjunct Functors

- For finite groups $M, L \subset G$,

$$\mathrm{Hom}_G\left(\pi, \mathrm{Ind}_M^G \sigma\right) \cong \mathrm{Hom}_M\left(\mathrm{Res}_M^G \pi, \sigma\right).$$

Composition of Adjunct Functors

- For finite groups $M, L \subset G$,

$$\mathrm{Hom}_G\left(\pi, \mathrm{Ind}_M^G\,\sigma\right) \cong \mathrm{Hom}_M\left(\mathrm{Res}_M^G\,\pi, \sigma\right).$$

- The composition is given by Mackey's Restriction Formula

$$\mathrm{Res}_L^G\,\mathrm{Ind}_M^G\,\sigma = \bigoplus_{w \in L\backslash G/M} \mathrm{Ind}_{L_w}^L\left(w \cdot \mathrm{Res}_{M_w}^M\,\sigma\right),$$

Composition of Adjunct Functors

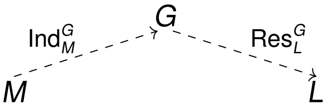
- For finite groups $M, L \subset G$,

$$\mathrm{Hom}_G\left(\pi, \mathrm{Ind}_M^G \sigma\right) \cong \mathrm{Hom}_M\left(\mathrm{Res}_M^G \pi, \sigma\right).$$

- The composition is given by Mackey's Restriction Formula

$$\mathrm{Res}_L^G \mathrm{Ind}_M^G \sigma = \bigoplus_{w \in L \backslash G / M} \mathrm{Ind}_{L_w}^L \left(w \cdot \mathrm{Res}_{M_w}^M \sigma\right),$$

where



Composition of Adjunct Functors

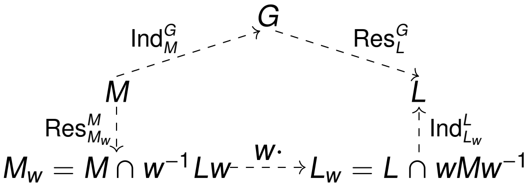
- For finite groups $M, L \subset G$,

$$\mathrm{Hom}_G\left(\pi, \mathrm{Ind}_M^G \sigma\right) \cong \mathrm{Hom}_M\left(\mathrm{Res}_M^G \pi, \sigma\right).$$

- The composition is given by Mackey's Restriction Formula

$$\mathrm{Res}_L^G \mathrm{Ind}_M^G \sigma = \bigoplus_{w \in L \backslash G / M} \mathrm{Ind}_{L_w}^L \left(w \cdot \mathrm{Res}_{M_w}^M \sigma\right),$$

where

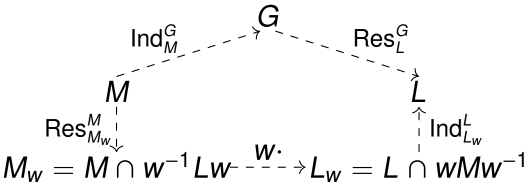


Composition of Adjunct Functors

- For G p-adic, $P, Q \subseteq G$ parabolic subgroups and $P = MU$ and $Q = LV$,
$$\mathrm{Hom}_G\left(\pi, \mathrm{Ind}_M^G \sigma\right) \cong \mathrm{Hom}_M\left(\mathrm{Res}_M^G \pi, \sigma\right).$$

- The composition is given by Mackey's Restriction Formula
$$\mathrm{Res}_L^G \, \mathrm{Ind}_M^G \sigma = \bigoplus_{w \in L \backslash G / M} \mathrm{Ind}_{L_w}^L \left(w \cdot \mathrm{Res}_{M_w}^M \sigma\right),$$

  where



Composition of Adjunct Functors

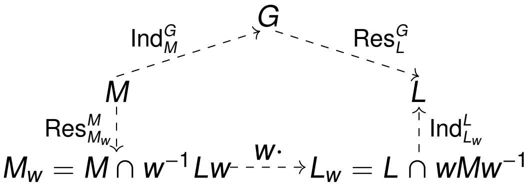
- For G p-adic, $P, Q \subseteq G$ parabolic subgroups and $P = MU$ and $Q = LV$,

$$\mathrm{Hom}_G\left(\pi, i_M^G\sigma\right) \cong \mathrm{Hom}_M\left(r_M^G\pi, \sigma\right).$$

- The composition is given by Mackey's Restriction Formula

$$\mathrm{Res}_L^G \,\mathrm{Ind}_M^G\, \sigma = \bigoplus_{w \in L\backslash G/M} \mathrm{Ind}_{L_w}^L \left(w \cdot \mathrm{Res}_{M_w}^M\, \sigma\right),$$

where



Composition of Adjunct Functors

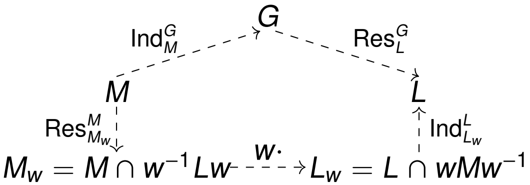
- For G p-adic, $P, Q \subseteq G$ parabolic subgroups and $P = MU$ and $Q = LV$,

$$\mathrm{Hom}_G\left(\pi, i_M^G \sigma\right) \cong \mathrm{Hom}_M\left(r_M^G \pi, \sigma\right).$$

- The composition is given by the geometric lemma

$$\mathrm{Res}_L^G \, \mathrm{Ind}_M^G \, \sigma = \bigoplus_{w \in L \backslash G / M} \mathrm{Ind}_{L_w}^L \left(w \cdot \mathrm{Res}_{M_w}^M \, \sigma\right),$$

where



Composition of Adjunct Functors

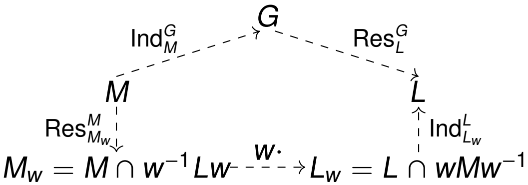
- For G p-adic, $P, Q \subseteq G$ parabolic subgroups and $P = MU$ and $Q = LV$,

$$\mathrm{Hom}_G\left(\pi, i_M^G \sigma\right) \cong \mathrm{Hom}_M\left(r_M^G \pi, \sigma\right).$$

- The composition is given by the geometric lemma

$$r_L^G i_M^G \sigma = \bigoplus_{w \in L \backslash G / M} i_{L_w}^L \left(w \cdot r_{M_w}^M \sigma\right),$$

where



Composition of Adjunct Functors

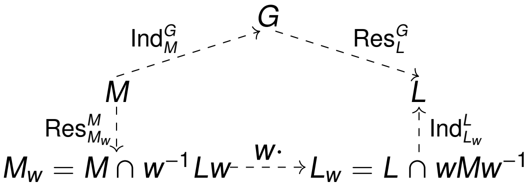
- For G p-adic, $P, Q \subseteq G$ parabolic subgroups and $P = MU$ and $Q = LV$,

$$\mathrm{Hom}_G\left(\pi, i_M^G \sigma\right) \cong \mathrm{Hom}_M\left(r_M^G \pi, \sigma\right).$$

- The composition is given by the geometric lemma

$$s.s.\left(r_L^G i_M^G \sigma\right) = \sum_{w \in L\backslash G/M} i_{L_w}^L\left(w \cdot r_{M_w}^M \sigma\right),$$

where



Composition of Adjunct Functors

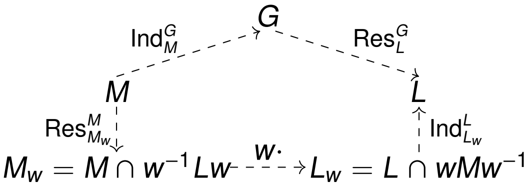
- For G p-adic, $P, Q \subseteq G$ parabolic subgroups and $P = MU$ and $Q = LV$,

$$\mathrm{Hom}_G\left(\pi, i_M^G\sigma\right) \cong \mathrm{Hom}_M\left(r_M^G\pi, \sigma\right).$$

- The composition is given by the geometric lemma

$$s.s.\left(r_L^G i_M^G\sigma\right) = \sum_{w\in Q\backslash G/P} i_{L_w}^L\left(w \cdot r_{M_w}^M\sigma\right),$$

where



Composition of Adjunct Functors

- For G p-adic, $P, Q \subseteq G$ parabolic subgroups and $P = MU$ and $Q = LV$,

$$\mathrm{Hom}_G\left(\pi, i_M^G \sigma\right) \cong \mathrm{Hom}_M\left(r_M^G \pi, \sigma\right).$$

- The composition is given by the geometric lemma

$$s.s.\left(r_L^G i_M^G \sigma\right) = \sum_{w \in Q \backslash G / P} i_{L_w}^L\left(w \cdot r_{M_w}^M \sigma\right),$$

where

$$M_w = M \cap w^{-1} L w, \quad L_w = w M w^{-1} \cap L.$$

Composition of Adjunct Functors

- For G p-adic, $P, Q \subseteq G$ parabolic subgroups and $P = MU$ and $Q = LV$,

$$\mathrm{Hom}_G\left(\pi, i_M^G\sigma\right) \cong \mathrm{Hom}_M\left(r_M^G\pi, \sigma\right).$$

- The composition is given by the geometric lemma

$$s.s.\left(r_L^G i_M^G\sigma\right) = \sum_{w\in Q\backslash G/P} i_{L_w}^L\left(w \cdot r_{M_w}^M\sigma\right),$$

where

$$M_w = M \cap w^{-1}Lw, \quad L_w = wMw^{-1} \cap L.$$

- Note that $Q\backslash G/P$ is a finite set.

Composition of Adjunct Functors

- For G p-adic, $P, Q \subseteq G$ parabolic subgroups and $P = MU$ and $Q = LV$,
$$\mathrm{Hom}_G\left(\pi, i_M^G\sigma\right) \cong \mathrm{Hom}_M\left(r_M^G\pi, \sigma\right).$$

- The composition is given by the geometric lemma
$$s.s.\left(r_L^G i_M^G\sigma\right) = \sum_{w\in Q\backslash G/P} i_{L_w}^L\left(w\cdot r_{M_w}^M\sigma\right),$$
where
$$M_w = M\cap w^{-1}Lw, \quad L_w = wMw^{-1}\cap L.$$

- Note that $Q\backslash G/P$ is a finite set.
- If $L = T$ and $\lambda_0 = r_T^M\Omega$, then
$$\boxed{r_T^G\left(i_M^G\Omega\right) = \sum_{w\in B\backslash G/P} w\cdot\lambda_0.}$$

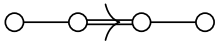
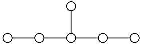
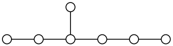
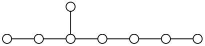
Degenerate Principle Series of Exceptional Groups

Degenerate Principal Series of Exceptional Groups

If M is maximal and Ω is 1-dimensional, $i_M^G \Omega$ is called a **degenerate principal series**.

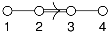
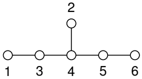
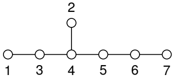
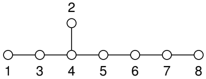
Degenerate Principal Series of Exceptional Groups

If M is maximal and Ω is 1-dimensional, $i_M^G \Omega$ is called a **degenerate principal series**.

Type	Dyinkin diagram	$\lvert B\backslash G/B\rvert$	$\left(\left\lvert B\backslash G/P_{\Delta\backslash\{\alpha_i\}}\right\rvert\right)_i$
G_2		12	(6; 6)
F_4		1,152	(24; 96; 96; 24)
E_6		51,840	(27; 72; 216; 720; 216; 27)
E_7		2,903,040	(126; 576; 2,016; 10,080; 4,032; 1,512; 56)
E_8		696,729,600	(2,160; 17,280; 69,120; 483,840; 241,920; 60,480; 6,720; 240)

Degenerate Principal Series of Exceptional Groups

If M is maximal and Ω is 1-dimensional, $i_M^G \Omega$ is called a **degenerate principal series**.

Type	Dyinkin diagram	$\lvert B\backslash G/B \rvert$	$\left(\left\lvert B\backslash G/P_{\Delta\setminus\{\alpha_i\}}\right\rvert\right)_i$
G_2		12	(6; 6)
F_4		1,152	(24; 96; 96; 24)
E_6		51,840	(27; 72; 216; 720; 216; 27)
E_7		2,903,040	(126; 576; 2,016; 10,080; 4,032; 1,512; 56)
E_8		696,729,600	(2,160; 17,280; 69,120; 483,840; 241,920; 60,480; 6,720; 240)

Study of the degenerate principal series of exceptional groups:
G_2 - Muić. F_4 - Choi, Jantzen. E_6 - Halawi, S., E_7, E_8 - a work in progress.

Branching Rules

Branching Rules in the Service of Humanity

- Fix M maximal and 1-dimensional Ω and $\pi = i_M^G \Omega$.

Branching Rules in the Service of Humanity

- Fix M maximal and 1-dimensional Ω and $\pi = i_M^G \Omega$.
- We wish to test if π is irreducible.

Branching Rules in the Service of Humanity

- Fix M maximal and 1-dimensional Ω and $\pi = i_M^G \Omega$.
- We wish to test if π is irreducible.
- Let $\lambda_0 = r_T^M \Omega$ and fix $\lambda_{init} \leq r_T^G \left(i_M^G \Omega \right)$.

Branching Rules in the Service of Humanity

- Fix M maximal and 1-dimensional Ω and $\pi = i_M^G \Omega$.
- We wish to test if π is irreducible.
- Let $\lambda_0 = r_T^M \Omega$ and fix $\lambda_{init} \leq r_T^G \left(i_M^G \Omega \right)$.
- Fix an irreducible subquotient $\pi_0 \leq \pi$ such that $\lambda_{init} \leq r_T^G \pi_0$.

Branching Rules in the Service of Humanity

- Fix M maximal and 1-dimensional Ω and $\pi = i_M^G \Omega$.
- We wish to test if π is irreducible.
- Let $\lambda_0 = r_T^M \Omega$ and fix $\lambda_{init} \le r_T^G \left(i_M^G \Omega \right)$.
- Fix an irreducible subquotient $\pi_0 \le \pi$ such that $\lambda_{init} \le r_T^G \pi_0$.
- Goal: prove that

$$r_T^G \pi_0 = r_T^G \pi.$$

Branching Rules in the Service of Humanity

- Fix M maximal and 1-dimensional Ω and $\pi = i_M^G \Omega$.
- We wish to test if π is irreducible.
- Let $\lambda_0 = r_T^M \Omega$ and fix $\lambda_{init} \le r_T^G \left(i_M^G \Omega \right)$.
- Fix an irreducible subquotient $\pi_0 \le \pi$ such that $\lambda_{init} \le r_T^G \pi_0$.
- Goal: prove that
$$r_T^G \pi_0 = r_T^G \pi.$$
- This would imply that $\pi = \pi_0$ is irreducible.

Branching Rules in the Service of Humanity

- Fix M maximal and 1-dimensional Ω and $\pi = i_M^G \Omega$.
- We wish to test if π is irreducible.
- Let $\lambda_0 = r_T^M \Omega$ and fix $\lambda_{init} \leq r_T^G \left(i_M^G \Omega \right)$.
- Fix an irreducible subquotient $\pi_0 \leq \pi$ such that $\lambda_{init} \leq r_T^G \pi_0$.
- Goal: prove that
$$r_T^G \pi_0 = r_T^G \pi.$$
- This would imply that $\pi = \pi_0$ is irreducible.
- We use "branching rules" to prove that $r_T^G \pi_0 = r_T^G \pi$.

Branching Rules in the Service of Humanity

- Fix M maximal and 1-dimensional Ω and $\pi = i_M^G \Omega$.
- We wish to test if π is irreducible.
- Let $\lambda_0 = r_T^M \Omega$ and fix $\lambda_{init} \leq r_T^G \left(i_M^G \Omega \right)$.
- Fix an irreducible subquotient $\pi_0 \leq \pi$ such that $\lambda_{init} \leq r_T^G \pi_0$.
- Goal: prove that
$$r_T^G \pi_0 = r_T^G \pi.$$
- This would imply that $\pi = \pi_0$ is irreducible.
- We use "branching rules" to prove that $r_T^G \pi_0 = r_T^G \pi$.
- Aside from proving the irreducibility of $\pi = i_M^G \Omega$, the procedure described bellow can also be used to study the composition factors of a reducible π.

What is a Branching Rule?

A branching rule is an inference of the type:

$$\lambda_1 \leq r_T^G \pi_0 \Rightarrow \sum_{i=1}^{k} n_i \lambda_i \leq r_T^G \pi_0.$$

What is a Branching Rule?

A branching rule is an inference of the type:

$$\lambda_1 \leq r_T^G \pi_0 \Rightarrow \sum_{i=1}^{k} n_i \lambda_i \leq r_T^G \pi_0.$$

How it works (simple version)

- One finds a Levi L with a unique irreducible representation σ of L such that $\lambda_1 \leq r_T^L \sigma$.

What is a Branching Rule?

A branching rule is an inference of the type:

$$\lambda_1 \leq r^G_T \pi_0 \Rightarrow \sum_{i=1}^{k} n_i \lambda_i \leq r^G_T \pi_0.$$

How it works (simple version)

- One finds a Levi L with a unique irreducible representation σ of L such that $\lambda_1 \leq r^L_T \sigma$. Necessarily, $\sigma \leq r^G_L \pi_0$.

What is a Branching Rule?

A branching rule is an inference of the type:

$$\lambda_1 \leq r_T^G \pi_0 \Rightarrow \sum_{i=1}^{k} n_i \lambda_i \leq r_T^G \pi_0.$$

How it works (simple version)

- One finds a Levi L with a unique irreducible representation σ of L such that $\lambda_1 \leq r_T^L \sigma$. Necessarily, $\sigma \leq r_L^G \pi_0$.
- By taking Jacquet functor in stages,

$$r_T^L \sigma \leq r_T^L r_L^G \pi_0 = r_T^G \pi_0.$$

What is a Branching Rule?

A branching rule is an inference of the type:

$$\lambda_1 \leq r_T^G \pi_0 \Rightarrow \sum_{i=1}^{k} n_i \lambda_i \leq r_T^G \pi_0.$$

How it works (simple version)

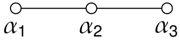
- One finds a Levi L with a unique irreducible representation σ of L such that $\lambda_1 \leq r_T^L \sigma$. Necessarily, $\sigma \leq r_L^G \pi_0$.
- By taking Jacquet functor in stages,

$$r_T^L \sigma \leq r_T^L r_L^G \pi_0 = r_T^G \pi_0.$$

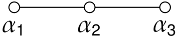
Our aim is, by successive application of various branching rules, to show that

$$\lambda_0 \leq r_T^G \pi_0 \Rightarrow ... \Rightarrow r_T^G \pi = \sum_{w \in B \backslash G / P} w \cdot \lambda \leq r_T^G \pi_0.$$

Example: $G = SL_4\left(\mathbb{Q}_p\right)$ (type A_3)

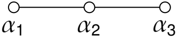


Example: $G = SL_4\left(\mathbb{Q}_p\right)$ (type A_3)



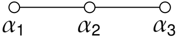
$$B = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad T = \left\{ t = \begin{pmatrix} t_1 & \color{blue}{\alpha_1} & & \\ & \frac{t_2}{t_1} & \color{blue}{\alpha_2} & \\ & & \frac{t_3}{t_2} & \color{blue}{\alpha_3} \\ & & & \frac{1}{t_3} \end{pmatrix} \right\}$$

Example: $G = SL_4\left(\mathbb{Q}_p\right)$ (type A_3)



$$B = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad T = \left\{ t = \begin{pmatrix} t_1 & \alpha_1 & & \\ & \frac{t_2}{t_1} & \alpha_2 & \\ & & \frac{t_3}{t_2} & \alpha_3 \\ & & & \frac{1}{t_3} \end{pmatrix} \right\}, \quad \omega_i(t) = t_i$$

Example: $G = SL_4\left(\mathbb{Q}_p\right)$ (type A_3)



$$B = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad T = \left\{ t = \begin{pmatrix} t_1 & \color{blue}{\alpha_1} & & \\ & \frac{t_2}{t_1} & \color{blue}{\alpha_2} & \\ & & \frac{t_3}{t_2} & \color{blue}{\alpha_3} \\ & & & \frac{1}{t_3} \end{pmatrix} \right\}, \quad \omega_i\left(t\right) = t_i$$

$$P_1 = \left(\begin{array}{cc|c|c} * & * & * & * \\ * & * & * & * \\ \hline 0 & 0 & * & * \\ \hline 0 & 0 & 0 & * \end{array}\right), \quad P_2 = \left(\begin{array}{c|cc|c} * & * & * & * \\ \hline 0 & * & * & * \\ 0 & * & * & * \\ \hline 0 & 0 & 0 & * \end{array}\right), \quad P_3 = \left(\begin{array}{c|c|cc} * & * & * & * \\ \hline 0 & * & * & * \\ \hline 0 & 0 & * & * \\ 0 & 0 & * & * \end{array}\right)$$

$$P_{1,2} = \left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ \hline 0 & 0 & 0 & * \end{array}\right), \quad P_{1,3} = \left(\begin{array}{cc|cc} * & * & * & * \\ * & * & * & * \\ \hline 0 & 0 & * & * \\ 0 & 0 & * & * \end{array}\right), \quad P_{2,3} = \left(\begin{array}{c|ccc} * & * & * & * \\ \hline 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right)$$

Example: $G = SL_4\left(\mathbb{Q}_p\right)$ (type A_3)

$$\underset{\alpha_1}{\circ}\!\!-\!\!-\!\!-\!\!\underset{\alpha_2}{\circ}\!\!-\!\!-\!\!-\!\!\underset{\alpha_3}{\circ}$$

$$B = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad T = \left\{ t = \begin{pmatrix} t_1 & \color{blue}{\alpha_1} & & \\ & \frac{t_2}{t_1} & \color{blue}{\alpha_2} & \\ & & \frac{t_3}{t_2} & \color{blue}{\alpha_3} \\ & & & \frac{1}{t_3} \end{pmatrix} \right\}, \quad \omega_i(t) = t_i$$

$$P_1 = \left(\begin{array}{cc|c|c} * & * & * & * \\ * & * & * & * \\ \hline 0 & 0 & * & * \\ \hline 0 & 0 & 0 & * \end{array}\right), \quad P_2 = \left(\begin{array}{c|cc|c} * & * & * & * \\ \hline 0 & * & * & * \\ 0 & * & * & * \\ \hline 0 & 0 & 0 & * \end{array}\right), \quad P_3 = \left(\begin{array}{c|c|cc} * & * & * & * \\ \hline 0 & * & * & * \\ \hline 0 & 0 & * & * \\ 0 & 0 & * & * \end{array}\right)$$

$$P_{1,2} = \left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ \hline 0 & 0 & 0 & * \end{array}\right), \quad P_{1,3} = \left(\begin{array}{cc|cc} * & * & * & * \\ * & * & * & * \\ \hline 0 & 0 & * & * \\ 0 & 0 & * & * \end{array}\right), \quad P_{2,3} = \left(\begin{array}{c|ccc} * & * & * & * \\ \hline 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right)$$

$$\lambda = (\lambda_1, \lambda_2, \lambda_3) \rightsquigarrow (\lambda_1, \lambda_2, \lambda_3)(t) = |t_1|^{\lambda_1} \cdot |t_2|^{\lambda_2} \cdot |t_3|^{\lambda_3}$$

Example in $SL_4\left(\mathbb{Q}_p\right)$

- We prove that $\pi = i_{M_{2,3}}^{G}\,\mathbf{1}_{M_{2,3}}$ is irreducible.

Example in $SL_4(\mathbb{Q}_p)$

- We prove that $\pi = i_{M_{2,3}}^{G}\mathbf{1}_{M_{2,3}}$ is irreducible.
- $\lambda_0 = r_{T}^{M_{2,3}}\mathbf{1}_{M_{2,3}} = (1,-1,-1)$.

λ	$(-1,0,-1)$	$(1,-1,-1)$	$(-1,-1,1)$
$r_T^G\pi$	2	1	1

Example in $SL_4\left(\mathbb{Q}_p\right)$

- We prove that $\pi = i_{M_{2,3}}^{G} \mathbf{1}_{M_{2,3}}$ is irreducible.
- $\lambda_0 = r_{T}^{M_{2,3}} \mathbf{1}_{M_{2,3}} = (1,-1,-1)$.
- Let $\pi_0 \leq \pi$ be irreducible, such that $(-1,0,-1) \leq r_T^G \pi_0$.

λ	$(-1,0,-1)$	$(1,-1,-1)$	$(-1,-1,1)$
$r_T^G \pi$	2	1	1

Example in $SL_4(\mathbb{Q}_p)$

- We prove that $\pi = i_{M_{2,3}}^{G} \mathbf{1}_{M_{2,3}}$ is irreducible.
- $\lambda_0 = r_{T}^{M_{2,3}} \mathbf{1}_{M_{2,3}} = (1,-1,-1)$.
- Let $\pi_0 \leq \pi$ be irreducible, such that $(-1,0,-1) \leq r_T^G \pi_0$.

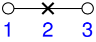
λ	$(-1,0,-1)$	$(1,-1,-1)$	$(-1,-1,1)$
$r_T^G \pi$	2	1	1
$r_T^G \pi_0 \geq$	1	0	0

Example in $SL_4\left(\mathbb{Q}_p\right)$

- We prove that $\pi = i^G_{M_{2,3}}\,\mathbf{1}_{M_{2,3}}$ is irreducible.
- $\lambda_0 = r^{M_{2,3}}_T\,\mathbf{1}_{M_{2,3}} = (1,-1,-1)$.
- Let $\pi_0 \leq \pi$ be irreducible, such that $(-1,0,-1) \leq r^G_T\pi_0$.

$i^{M_2}_T(-1,0,-1) = \left(i^{M_2}_T\,\mathbf{1}_T\right) \otimes (-1,0,-1)$ is the unique irreducible representation of M_2 containing $(-1,0,-1)$.

$$r^G_T\left(i^{M_2}_T(-1,0,-1)\right) = 2 \times (-1,0,-1).$$



$$T = \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{pmatrix} \qquad M_2 = \begin{pmatrix} * & & & \\ & * & * & \\ & * & * & \\ & & & * \end{pmatrix}$$

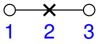
λ	$(-1,0,-1)$	$(1,-1,-1)$	$(-1,-1,1)$
$r^G_T\pi$	2	1	1
$r^G_T\pi_0 \geq$	1	0	0

Example in $SL_4\left(\mathbb{Q}_p\right)$

- We prove that $\pi = i_{M_{2,3}}^{G} \mathbf{1}_{M_{2,3}}$ is irreducible.
- $\lambda_0 = r_T^{M_{2,3}} \mathbf{1}_{M_{2,3}} = (1,-1,-1)$.
- Let $\pi_0 \leq \pi$ be unique irreducible, such that $(-1,0,-1) \leq r_T^G \pi_0$.

$i_T^{M_2}(-1,0,-1) = \left(i_T^{M_2} \mathbf{1}_T\right) \otimes (-1,0,-1)$ is the unique irreducible representation of M_2 containing $(-1,0,-1)$.

$$r_T^G\left(i_T^{M_2}(-1,0,-1)\right) = 2 \times (-1,0,-1).$$



$$T = \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{pmatrix} \qquad M_2 = \begin{pmatrix} * & & & \\ & * & * & \\ & * & * & \\ & & & * \end{pmatrix}$$

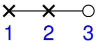
λ	$(-1,0,-1)$	$(1,-1,-1)$	$(-1,-1,1)$
$r_T^G \pi$	2	1	1
$r_T^G \pi_0 \geq$	2	0	0

Example in $SL_4\left(\mathbb{Q}_p\right)$

- We prove that $\pi = i_{M_{2,3}}^{G} \mathbf{1}_{M_{2,3}}$ is irreducible.
- $\lambda_0 = r_T^{M_{2,3}} \mathbf{1}_{M_{2,3}} = (1, -1, -1)$.
- Let $\pi_0 \leq \pi$ be unique irreducible, such that $(-1, 0, -1) \leq r_T^G \pi_0$.

$i_{M_1}^{M_{1,2}} det_{M_1}^{-1/2}$ is the unique irreducible representation of $M_{1,2}$ containing $(-1, 0, -1)$.

$$r_T^G\left(i_{M_1}^{M_{1,2}} det_{M_1}^{-1/2}\right) = 2 \times (-1, 0, -1) + (1, -1, -1).$$



$$M_1 = \left(\begin{array}{cc|c|c} * & * & & \\ * & * & & \\ \hline & & * & \\ \hline & & & * \end{array}\right) \qquad M_{1,2} = \left(\begin{array}{ccc|c} * & * & * & \\ * & * & * & \\ * & * & * & \\ \hline & & & * \end{array}\right)$$

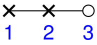
λ	$(-1, 0, -1)$	$(1, -1, -1)$	$(-1, -1, 1)$
$r_T^G \pi$	2	1	1
$r_T^G \pi_0 \geq$	2	0	0

Example in $SL_4(\mathbb{Q}_p)$

- We prove that $\pi = i_{M_{2,3}}^{G}\mathbf{1}_{M_{2,3}}$ is irreducible.
- $\lambda_0 = r_T^{M_{2,3}}\mathbf{1}_{M_{2,3}} = (1,-1,-1)$.
- Let $\pi_0 \le \pi$ be unique irreducible, such that $(-1,0,-1) \le r_T^G \pi_0$.

$i_{M_1}^{M_{1,2}} det_{M_1}^{-1/2}$ is the unique irreducible representation of $M_{1,2}$ containing $(-1,0,-1)$.

$$r_T^G\left(i_{M_1}^{M_{1,2}} det_{M_1}^{-1/2}\right) = 2\times(-1,0,-1)+(1,-1,-1).$$



$$M_1 = \left(\begin{array}{cc|c|c} * & * & & \\ * & * & & \\ \hline & & * & \\ \hline & & & * \end{array}\right) \qquad M_{1,2} = \left(\begin{array}{ccc|c} * & * & * & \\ * & * & * & \\ * & * & * & \\ \hline & & & * \end{array}\right)$$

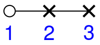
λ	$(-1,0,-1)$	$(1,-1,-1)$	$(-1,-1,1)$
$r_T^G\pi$	2	1	1
$r_T^G\pi_0 \ge$	2	1	0

Example in $SL_4\left(\mathbb{Q}_p\right)$

- We prove that $\pi = i_{M_{2,3}}^{G} \mathbf{1}_{M_{2,3}}$ is irreducible.
- $\lambda_0 = r_{T}^{M_{2,3}} \mathbf{1}_{M_{2,3}} = (1,-1,-1)$.
- Let $\pi_0 \leq \pi$ be unique irreducible, such that $(-1,0,-1) \leq r_T^G \pi_0$.

$i_{M_3}^{M_{2,3}} det_{M_3}^{-1/2}$ is the unique irreducible representation of $M_{2,3}$ containing $(-1,0,-1)$.

$$r_T^G\left(i_{M_3}^{M_{2,3}} det_{M_3}^{-1/2}\right) = 2 \times (-1,0,-1) + (-1,-1,1).$$



$$M_3 = \left(\begin{array}{c|c|cc} * & & & \\ \hline & * & & \\ \hline & & * & * \\ & & * & * \end{array}\right) \qquad M_{2,3} = \left(\begin{array}{c|ccc} * & & & \\ \hline & * & * & * \\ & * & * & * \\ & * & * & * \end{array}\right)$$

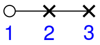
λ	$(-1,0,-1)$	$(1,-1,-1)$	$(-1,-1,1)$
$r_T^G \pi$	2	1	1
$r_T^G \pi_0 \geq$	2	1	0

Example in $SL_4(\mathbb{Q}_p)$

- We prove that $\pi = i_{M_{2,3}}^{G} \mathbf{1}_{M_{2,3}}$ is irreducible.
- $\lambda_0 = r_T^{M_{2,3}} \mathbf{1}_{M_{2,3}} = (1,-1,-1)$.
- Let $\pi_0 \leq \pi$ be unique irreducible, such that $(-1,0,-1) \leq r_T^G \pi_0$.

$i_{M_3}^{M_{2,3}} det_{M_3}^{-1/2}$ is the unique irreducible representation of $M_{2,3}$ containing $(-1,0,-1)$.

$$r_T^G \left(i_{M_3}^{M_{2,3}} det_{M_3}^{-1/2} \right) = 2 \times (-1,0,-1) + (-1,-1,1).$$



$$M_3 = \left(\begin{array}{c|c|cc} * & & & \\ \hline & * & & \\ \hline & & * & * \\ & & * & * \end{array} \right) \quad M_{2,3} = \left(\begin{array}{c|ccc} * & & & \\ \hline & * & * & * \\ & * & * & * \\ & * & * & * \end{array} \right)$$

λ	$(-1,0,-1)$	$(1,-1,-1)$	$(-1,-1,1)$
$r_T^G \pi$	2	1	1
$r_T^G \pi_0 \geq$	2	1	1

Example in $SL_4\left(\mathbb{Q}_p\right)$

- We prove that $\pi = i^G_{M_{2,3}} \mathbf{1}_{M_{2,3}}$ is irreducible.
- $\lambda_0 = r^{M_{2,3}}_T \mathbf{1}_{M_{2,3}} = (1,-1,-1)$.
- Let $\pi_0 \leq \pi$ be unique irreducible, such that $(-1,0,-1) \leq r^G_T \pi_0$.

- We proved that $r^G_T \pi_0 = r^G_T \pi$.

λ	$(-1,0,-1)$	$(1,-1,-1)$	$(-1,-1,1)$
$r^G_T \pi$	2	1	1
$r^G_T \pi_0 \geq$	2	1	1

Example in $SL_4\left(\mathbb{Q}_p\right)$

- We prove that $\pi = i_{M_{2,3}}^{G} \mathbf{1}_{M_{2,3}}$ is irreducible.
- $\lambda_0 = r_T^{M_{2,3}} \mathbf{1}_{M_{2,3}} = (1, -1, -1)$.
- Let $\pi_0 \leq \pi$ be unique irreducible, such that $(-1, 0, -1) \leq r_T^G \pi_0$.

- We proved that $r_T^G \pi_0 = r_T^G \pi$.

λ	$(-1, 0, -1)$	$(1, -1, -1)$	$(-1, -1, 1)$
$r_T^G \pi$	2	1	1
$r_T^G \pi_0 \geq$	2	1	1

- It follows that $\pi = \pi_0$ is irreducible.

Irreducible Subrepresentations

Irreducible Subrepresentations

Proposition

Let M be a standard Levi subgroup of G and let Ω be a 1-dimensional representation of M with $\lambda_0 = r_T^G \Omega$ and let $\pi = i_M^G \Omega$.

Irreducible Subrepresentations

Proposition

Let M be a standard Levi subgroup of G and let Ω be a 1-dimensional representation of M with $\lambda_0 = r_T^G \Omega$ and let $\pi = i_M^G \Omega$.

1. If $\pi' \hookrightarrow \pi$, then $\lambda_0 \leq r_T^G \pi'$.

Irreducible Subrepresentations

Proposition

Let M be a standard Levi subgroup of G and let Ω be a 1-dimensional representation of M with $\lambda_0 = r_T^G \Omega$ and let $\pi = i_M^G \Omega$.

1. If $\pi' \hookrightarrow \pi$, then $\lambda_0 \leq r_T^G \pi'$.
2. If there exists $\pi_0 \leq \pi$ such that

$$mult\left(\lambda_0, r_T^G \pi_0\right) = mult\left(\lambda_0, r_T^G \pi\right),$$

   then π_0 is the unique irreducible subrepresentation of π.

Irreducible Subrepresentations

Proposition

Let M be a standard Levi subgroup of G and let Ω be a 1-dimensional representation of M with $\lambda_0 = r_T^G \Omega$ and let $\pi = i_M^G \Omega$.

1. If $\pi' \hookrightarrow \pi$, then $\lambda_0 \leq r_T^G \pi'$.
2. If there exists $\pi_0 \leq \pi$ such that
$$mult\left(\lambda_0, r_T^G \pi_0\right) = mult\left(\lambda_0, r_T^G \pi\right),$$
then π_0 is the unique irreducible subrepresentation of π.

Proof

1. Since $\pi = i_M^G \Omega \hookrightarrow i_T^G \lambda_0$,

Irreducible Subrepresentations

Proposition

Let M be a standard Levi subgroup of G and let Ω be a 1-dimensional representation of M with $\lambda_0 = r_T^G \Omega$ and let $\pi = i_M^G \Omega$.

1. If $\pi' \hookrightarrow \pi$, then $\lambda_0 \leq r_T^G \pi'$.
2. If there exists $\pi_0 \leq \pi$ such that

$$mult\left(\lambda_0, r_T^G \pi_0\right) = mult\left(\lambda_0, r_T^G \pi\right),$$

   then π_0 is the unique irreducible subrepresentation of π.

Proof

1. Since $\pi = i_M^G \Omega \hookrightarrow i_T^G \lambda_0$, it follows that

$$0 \neq \mathrm{Hom}_G\left(\pi', i_T^G \lambda_0\right) = \mathrm{Hom}_T\left(r_T^G \pi', \lambda_0\right).$$

Irreducible Subrepresentations

Proposition

Let M be a standard Levi subgroup of G and let Ω be a 1-dimensional representation of M with $\lambda_0 = r_T^G \Omega$ and let $\pi = i_M^G \Omega$.

1. If $\pi' \hookrightarrow \pi$, then $\lambda_0 \leq r_T^G \pi'$.
2. If there exists $\pi_0 \leq \pi$ such that

$$mult\left(\lambda_0, r_T^G \pi_0\right) = mult\left(\lambda_0, r_T^G \pi\right),$$

   then π_0 is the unique irreducible subrepresentation of π.

Proof

1. Since $\pi = i_M^G \Omega \hookrightarrow i_T^G \lambda_0$, it follows that

$$0 \neq \mathrm{Hom}_G\left(\pi', i_T^G \lambda_0\right) = \mathrm{Hom}_T\left(r_T^G \pi', \lambda_0\right).$$

   Hence, $\lambda_0 \leq r_T^G \pi'$.

Irreducible Subrepresentations

Proposition

Let M be a standard Levi subgroup of G and let Ω be a 1-dimensional representation of M with $\lambda_0 = r_T^G \Omega$ and let $\pi = i_M^G \Omega$.

1. If $\pi' \hookrightarrow \pi$, then $\lambda_0 \leq r_T^G \pi'$.
2. If there exists $\pi_0 \leq \pi$ such that
$$mult\left(\lambda_0, r_T^G \pi_0\right) = mult\left(\lambda_0, r_T^G \pi\right),$$
then π_0 is the unique irreducible subrepresentation of π.

Proof

1. Since $\pi = i_M^G \Omega \hookrightarrow i_T^G \lambda_0$, it follows that
$$0 \neq \mathrm{Hom}_G\left(\pi', i_T^G \lambda_0\right) = \mathrm{Hom}_T\left(r_T^G \pi', \lambda_0\right).$$
Hence, $\lambda_0 \leq r_T^G \pi'$.
2. This follows from 1.

Example in $SL_4\left(\mathbb{Q}_p\right)$

- Let $\pi = i_{M_{1,3}}^{G} det_{M_{1,3}}^{-1}$.

Example in $SL_4\left(\mathbb{Q}_p\right)$

- Let $\pi = i_{M_{1,3}}^{G} det_{M_{1,3}}^{-1}$.
- Fact: π is reducible.

Example in $SL_4\left(\mathbb{Q}_p\right)$

- Let $\pi = i_{M_{1,3}}^{G} det_{M_{1,3}}^{-1}$.
- Fact: π is reducible.
- $\lambda_0 = r_T^{M_{1,3}}\Omega = (-1, 0, -1)$.

Example in $SL_4\left(\mathbb{Q}_p\right)$

- Let $\pi = i_{M_{1,3}}^{G} det_{M_{1,3}}^{-1}$.
- Fact: π is reducible.
- $\lambda_0 = r_T^{M_{1,3}}\Omega = (-1, 0, -1)$.
- We wish to prove that π admits a unique irreducible subrepresentation.

Example in $SL_4\left(\mathbb{Q}_p\right)$

- Let $\pi = i_{M_{1,3}}^G det_{M_{1,3}}^{-1}$.
- Fact: π is reducible.
- $\lambda_0 = r_T^{M_{1,3}}\Omega = (-1,0,-1)$.
- We wish to prove that π admits a unique irreducible subrepresentation.
- Let $\pi_0 \leq \pi$ satisfy $\lambda_0 \leq r_T^G \pi_0$. We get

λ	$(-1,0,-1)$	$(1,-1,-1)$	$(-1,-1,1)$	$(-1,2,-1)$	$(1,-2,1)$
$r_T^G\pi$	2	1	1	1	1
$r_T^G\pi_0 \geq$	2	1	1	0	0

Example in $SL_4(\mathbb{Q}_p)$

- Let $\pi = i^G_{M_{1,3}} det^{-1}_{M_{1,3}}$.
- <u>Fact:</u> π is reducible.
- $\lambda_0 = r^{M_{1,3}}_T \Omega = (-1, 0, -1)$.
- We wish to prove that π admits a unique irreducible subrepresentation.
- Let $\pi_0 \leq \pi$ satisfy $\lambda_0 \leq r^G_T \pi_0$. We get

λ	$(-1,0,-1)$	$(1,-1,-1)$	$(-1,-1,1)$	$(-1,2,-1)$	$(1,-2,1)$
$r^G_T \pi$	2	1	1	1	1
$r^G_T \pi_0 \geq$	2	1	1	0	0

- It follows that π_0 is the unique irreducible subrepresentation of π.

Example in $SL_4\left(\mathbb{Q}_p\right)$

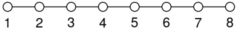
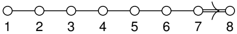
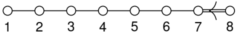
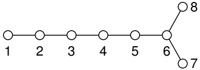
- Let $\pi = i^G_{M_{1,3}} det^{-1}_{M_{1,3}}$.
- Fact: π is reducible.
- $\lambda_0 = r^{M_{1,3}}_T \Omega = (-1, 0, -1)$.
- We wish to prove that π admits a unique irreducible subrepresentation.
- Let $\pi_0 \leq \pi$ satisfy $\lambda_0 \leq r^G_T \pi_0$. We get

λ	$(-1,0,-1)$	$(1,-1,-1)$	$(-1,-1,1)$	$(-1,2,-1)$	$(1,-2,1)$
$r^G_T \pi$	2	1	1	1	1
$r^G_T \pi_0 \geq$	2	1	1	0	0

- It follows that π_0 is the unique irreducible subrepresentation of π.
- Note that $r^G_T \pi_0 = r^G_T i^G_{M_{2,3}} \mathbf{1}_{M_{2,3}}$. In fact, $\pi_0 = i^G_{M_{2,3}} \mathbf{1}_{M_{2,3}}$.

Thank You!

For comparison, the sizes of the relevant cosets for classical types:

Type	Dyinkin diagram	$\|B\backslash G/B\|$	$\|B\backslash G/P_{\Delta\setminus\{\alpha_k\}}\|$
A_n		$(n+1)!$	$\frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$
A_8		$362,880$	(9;36;84;126;126;84;36;9)
BC_n		$2^n n!$	$\frac{2^n n!}{k!2^{n-k}(n-k)!} = 2^k\binom{n}{k}$
BC_8		$10,321,920$	(16;112;448;1,120;1,792;1,792;1,024;256)
D_n		$2^{n-1}n!$	$\begin{cases} 2^k\binom{n}{k} & ,k \le n-3 \\ 2^{n-3}n(n-1), & k = n-2 \\ 2^{n-1}, & k = n-1, n \end{cases}$
D_8		$5,160,960$	(16;112;448;1,120;1,792;1,792;128;128)

Study of the groups of classical types:
GL_n - Bernstein-Zelevinsky. SL_n - Gelbart-Knapp, Tadić. Sp_{2n}, SO_m - Ban, Jantzen.