$\mathcal{L}$-Functions of Cuspidal Representations of G_2 and Their Poles

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Introduction

Notations

Some notations for this talk:

- $\mathcal{P} = \{\infty, 2, 3, 5, 7, ...\}$ - the set of primes.
- $\mathbb{A}$ - ring of adeles of $\mathbb{Q}$.
- Fix a non-trivial additive character $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^{\times}$.
- Throughout the talk S will denote a finite subset of $\mathcal{P}$, S varies but always assumed to contain ∞.

Irreducible Representations of Adelic Groups

- Let G be an algebraic group defined over $\mathbb{Q}$
- Let π be an admissible irreducible representations of $G(\mathbb{A})$
  $\rightsquigarrow \pi = \otimes'_{p\in\mathcal{P}}\pi_p$:
  - π_p is irreducible for all $p \in \mathcal{P}$
  - π_p is unramified for almost all p (at any $p \notin S$ for some S)

Objective I

Describe the spectral decomposition of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ under the action of $G(\mathbb{A})$. Roughly

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = L^2_{disc.} \oplus L^2_{cont.} = L^2_{cusp.} \oplus L^2_{res.} \oplus L^2_{cont.}$$

In this talk we wish to describe parts of $L^2_{cusp.}$ for the exceptional group of type G_2.

$\mathcal{L}$-functions of Irreducible Representations

- Fix $\rho : {}^L G \to GL_N(\mathbb{C})$.
- For π_v unramified we define a complex function $\mathcal{L}\left(s, \pi_p, \rho\right)$

$$\mathcal{L}\left(s, \pi_p, \rho\right) = \frac{1}{det\left(\mathbf{1} - \rho\left(t_{\pi_p}\right) p^{-s}\right)} \qquad s \in \mathbb{C},$$

  where t_{π_p} is the Satake parameter of π_p.
- Let

$$\mathcal{L}^S(s, \pi, \rho) = \prod_{p \notin S} \mathcal{L}\left(s, \pi_p, \rho\right) \quad \downdownarrows \ \Re\mathfrak{e}(s) \gg 0$$

  $\mathcal{L}^S(s, \pi, \rho)$ let us study π by means of analytic enquiries.

Conjecture (Langlands)

$\mathcal{L}^S(s, \pi, \rho)$ admits a meromorphic continuation to $\mathbb{C}$.

Objective II

Prove the meromorphic continuation of the standard $\mathcal{L}$-function of a cuspidal representation of G_2.

Rankin-Selberg Method

1. Cook up a nice adelic integral + unfolding $\mathcal{Z}(s,\dots)$. Common construction:
$$\mathcal{Z}(s,\varphi,\mathcal{E}) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi(g)\,\mathcal{E}(g,s)\,dg$$
$$= \text{... unfolding...} = \int_{X(\mathbb{A})} \mathcal{M}_\varphi(g)\,\mathcal{F}(g,s)\,dg$$
with $X(\mathbb{A}) = \prod' X(\mathbb{Q}_p)$, $\mathcal{M}_\varphi = \otimes' \mathcal{M}_{\varphi,p}$ and $\mathcal{F} = \otimes' \mathcal{F}_p$.
2. Factorizability $\mathcal{Z}(s,\dots) = \prod\limits_{p\in\mathcal{P}} \mathcal{Z}_p(s,\dots)$. In the example (for $\mathfrak{Re}(s) \gg 0$):
$$= \prod \int_{X(\mathbb{Q}_p)} \mathcal{M}_{\varphi,p}(g)\,\mathcal{F}_p(g,s)\,dg$$
3. Unramified calculation - for $p \notin S$ prove $\mathcal{Z}_p(s,\dots) = \mathcal{L}\left(s,\pi_p,\mathfrak{st}\right)$.
4. Deal with the places in S.

Examples

- Tate's thesis:

$$\int_{\mathbb{A}^\times} \chi(x) |x|^s \Phi(x) \frac{dx}{x}$$

  for the Hecke $\mathcal{L}$-function of a character χ.

- Godement-Jacquet:

$$\int_{GL_n(\mathbb{A})} \left\langle \pi(g) v, v^\vee \right\rangle |det\, g|^s \Phi(g)\, dg$$

  for the standard $\mathcal{L}$-function of GL_n.

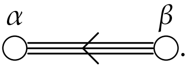
- Hecke:

$$\int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \varphi \begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s-\frac{1}{2}} d^\times a = \ldots = \int_{\mathbb{A}^\times} W_\varphi^\psi \begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s-\frac{1}{2}} d^\times a$$

  for the standard $\mathcal{L}$-function of GL_2.

The Exceptional Group of Type G_2

- Let $\mathbb{O}$ be the split Octonions over $\mathbb{Q}$.
- The group $G(F) = \mathrm{Aut}_F\left(\mathbb{O} \otimes_{\mathbb{Q}} F\right)$ is a group of type G_2.
- This gives a natural embedding of G in the split $SO(8)$ and in fact in $SO(7)$ (as the automorphism of imaginary octonions).
- G is a simple, simply-connected, adjoint group.
- B - Borel subgroup with torus T and unipotent radical N.



- $\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$
- P - Heisenberg parabolic subgroup, $P = M \cdot U$
- $M \cong GL_2$

Partial $\mathcal{L}$-Functions

- $\pi = \otimes_{p\in\mathcal{P}}\pi_p$ - irreducible cuspidal representation of $G(\mathbb{A})$.
- $\chi = \otimes_{p\in\mathcal{P}}\chi_p : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ - Hecke character.
- $S \subset \mathcal{P}$ - Finite set such that π_p and χ_p are unramified for $p \notin S$.
- For $p \notin S$ let $t_{\pi_p} \in {}^L G(\mathbb{C}) = G_2(\mathbb{C}) \times \mathrm{Gal}(\mathbb{Q}^{sep}/\mathbb{Q})$ be the Satake parameter of π_p.
- $\mathfrak{st}$ - Standard 7-dimensional representation of $G_2(\mathbb{C})$.
- $\mathcal{L}(s, \pi_p, \chi_p, \mathfrak{st}) = \left(\det\left(\mathbf{1} - \chi(p)\,\mathfrak{st}\left(t_{\pi_p}\right) p^{-s}\right)\right)^{-1}$
- $\mathcal{L}^S(s, \pi, \chi, \mathfrak{st}) = \prod_{p\notin S} \mathcal{L}(s, \pi_p, \chi_p, \mathfrak{st})$

Goal

1. Prove meromorphic continuation of $\mathcal{L}^S(s, \pi, \chi, \rho)$.
2. Study the poles of $\mathcal{L}^S(s, \pi, \chi, \rho)$.
3. Use $\mathcal{L}^S(s, \pi, \chi, \rho)$ to describe π using functorial lifts.

Étale Cubic Algebras

An étale cubic algebra E over a field F is one of the following

- $E = F \times F \times F$ - Split case
- $E = F \times K$ - K is a field
- E is a field

Surprising Correspondence (Gan-Gross-Savin 02', Huang-Magaard-Savin 98')

$$\left\{\begin{matrix}\text{Iso. classes of}\\ \text{Quasi-split}\\ \text{forms of } D_4/\mathbb{Q}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Iso. classes}\\ \text{of étale cubic}\\ \text{algebras over } \mathbb{Q}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Non-degenerate}\\ M(\mathbb{Q})\text{-orbits of}\\ \text{characters of } U(\mathbb{A})\end{matrix}\right\}$$

$$H_E \longleftrightarrow E \longleftrightarrow \Psi_E$$

Fourier Coefficients

Definition

$$L_{\Psi_E}(\varphi)(g) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\Psi_E(u)}du$$

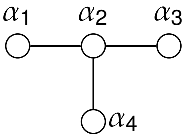
- We say π supports the Ψ_E-Fourier coefficient if $\exists\varphi \in \pi : \ L_{\Psi_E}(\varphi) \neq 0$.
- (Gan) For any π there exist at least one such E so that π supports the Ψ_E-Fourier coefficient.
- For any $g \in G(\mathbb{A})$ it holds that

$$L_{\Psi_E}(\cdot)(g) \in \mathrm{Hom}_{U(\mathbb{A})}\left(\pi, \mathbb{C}_{\Psi_E}\right).$$

- Let $L^2_{cusp.}(\Psi_E)$ denote the space of cuspidal representations of $G(\mathbb{A})$ supporting the Ψ_E-Fourier coefficient (not disjoint).
- Usually, $\dim_{\mathbb{C}}\left(\mathrm{Hom}_{U(\mathbb{A})}\left(\pi, \mathbb{C}_{\Psi_E}\right)\right) > 1$.

Quasi-Split Forms of D_4

- $H_E = Spin_8^E$ - The quasi-split form of $H = Spin_8$ associated with E.



- $1 \to H_E^{ad} \to \mathrm{Aut}\,(H_E) \to S_E \to 1$, where $S_E = \mathrm{Aut}_{\mathbb{Q}}\,(E)$. .
- $G = H_E^{S_E} \subset H_E$.
- $G \times S_E \subset H_E \rtimes S_E$ is a dual reductive pair.
- $B_E = T_E \cdot N_E$ with $B = B_E \cap G$
- $P_E = M_E \cdot U_E$ with $P = P_E \cap G$
- $M_E = \left\{ g \in \mathrm{Res}_{E/F}\, GL_2 \,\middle|\, \det(g) \in \mathbb{G}_m \right\}$. There is a natural map $\det_{M_E} : M_E \to \mathbb{G}_m$.

Global Theory

The Zeta Integral

- $I_{P_E}(\chi, s) = \mathrm{Ind}_{P_E(\mathbb{A})}^{H_E(\mathbb{A})}\left(\chi \circ \det_{M_E}\right) \otimes \left|\det_{M_E}\right|^{s+\frac{5}{2}}$.
- For the normalized[1] holomorphic section $f_s^* \in I_{P_E}(\chi, s)$ we define the normalized Eisenstein series

$$\mathcal{E}_E^*(\chi, s, f, h) = \sum_{\gamma \in P_E(\mathbb{Q}) \backslash H_E(\mathbb{Q})} f_s^*(\gamma h), \qquad h \in H_E(\mathbb{A}),\ \Downarrow \Re\mathfrak{e}(s) \gg 0.$$

- For $\varphi \in \pi$ let

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi(g)\, \mathcal{E}_E^*(\chi, s, f, g)\, dg.$$

This defines a meromorphic function for any φ, f.

[1] $f_s^*(1) = j_E(\chi, s)$ - a product of Dirichlet $\mathcal{L}$-functions

The Main Result

Theorem (S)

Assume that π supports the Ψ_E-Fourier coefficient and assume that $S \subset \mathcal{P}$ is a finite subset such that for any $p \notin S$ all data is unramified. It holds that

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \mathcal{L}^S(s, \pi, \chi, \rho)\, d_S(\chi, s, \Psi_E, \varphi_S, f_S).$$

Moreover, for any s_0 data can be chosen so that $d_S(\chi, s, \Psi_E, \varphi_S, f_S)$ is analytic and non-vanishing in neighborhood of s_0.

Corollary

$\mathcal{L}^S(s, \pi, \chi, \rho)$ admits a meromorphic continuation.

Unfolding

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi(g)\, \mathcal{E}_E^*(\chi, s, f, g)\, dg =$$

$$= \sum_{\mu \in P_E(\mathbb{Q})\backslash H_E(\mathbb{Q})/G(\mathbb{Q})} \int\limits_{\mathrm{Stab}_{G(\mathbb{Q})}(\mu)\backslash G(\mathbb{A})} \varphi(g)\, f_s(\mu g)\, dg.$$

Problem

Describe $P_E(\mathbb{Q})\backslash H_E(\mathbb{Q})/G(\mathbb{Q})$.

Theorem

$P_E(\mathbb{Q})\backslash H_E(\mathbb{Q})$ admits a unique open (and dense) $G(\mathbb{Q})$-orbit. One can choose a representative $\mu_E \in G(\mathbb{A})$ of this orbit whose stabilizer is $T_{3\alpha+2\beta} \cdot \ker \Psi_E$.

The stabilizer of a representative of any other orbit contains a unipotent radical of a parabolic subgroup of G.

Factorization of $\mathcal{Z}_E(\chi, s, \varphi, f)$

We fix a representative $\mu \in H_E(F)$ for the open orbit

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \int_{\mathrm{Stab}_{G(\mathbb{Q})}(\mu)\backslash G(\mathbb{A})} \varphi(g) f_s(\mu_E g)\, dg = \int_{U(\mathbb{A})\backslash G(\mathbb{A})} L_{\Psi_E}(\varphi)(g) F^*(\Psi_E, \chi, g, s)\, dg$$

Obstacle

As (usually) $\dim \mathrm{Hom}_{U(\mathbb{A})}(\pi, \Psi_E) > 1$, the integral is not factorizable in the usual sense.

(Refined) Unramified Calculation

For $\nu \notin S$ let v_0 be a spherical vector in π_ν. There exists $s_0 \in \mathbb{R}$ such that for any $\Re\mathfrak{e}(s) > s_0$ and **any** $\Lambda \in \mathrm{Hom}_{U(F_\nu)}\left(\pi_\nu, \mathbb{C}_{\Psi_{E,\nu}}\right)$ it holds that

$$\int_{U(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} F^*_\nu(\Psi_{E,\nu}, \chi_\nu, g, s)\, \Lambda(\pi_\nu(g) v_0)\, dg = \mathcal{L}\left(s + \frac{1}{2}, \pi_\nu, \chi_\nu, \mathfrak{st}\right) \Lambda(v_0)$$

Inductive Process

- $\mathcal{Z}_E(\chi, s, \varphi, f) = \lim\limits_{\substack{\longrightarrow \\ S\subset\Omega\subset\mathcal{P},\ |\Omega|<\infty}} \int_{X(\mathbb{A}_\Omega)} L_{\Psi_E}(\varphi)(g) F^*_\Omega(g, s)\, dg$
- $\int_{X(\mathbb{A}_{\Omega\cup\{p\}})} L_{\Psi_E}(\varphi)(g) F^*_{\Omega\cup\{p\}}(g, s)\ dg$

$$= \int_{X(\mathbb{A}_\Omega)} F^*_\Omega(g, s) \int_{X(\mathbb{Q}_p)} L_{\Psi_E}(\varphi)\left(gg_p\right) F^*_\nu\left(g_p, s\right)\ dg_p\ dg$$

$$= \mathcal{L}\left(s + \frac{1}{2}, \pi_p, \chi_p, \mathfrak{st}\right) \int_{X(\mathbb{A}_\Omega)} L_{\Psi_E}(\varphi)(g) F^*_\Omega(g, s)\ dg$$

$$\int_{X(\mathbb{A}_\Omega)} \cdots dg = \left(\prod_{p\in\Omega\setminus S} \mathcal{L}\left(s + \frac{1}{2}, \pi_p, \chi_p, \mathfrak{st}\right)\right) \int_{X(\mathbb{A}_S)} \cdots dg$$

- $\mathcal{Z}_E(\chi, s, \varphi, f) = \mathcal{L}^S\left(s + \frac{1}{2}, \pi, \chi, \mathfrak{st}\right) \int_{X(\mathbb{A}_S)} L_{\Psi_E}(\varphi)(g) F^*_S(g, s)\, dg$

$X(\cdot) := U(\cdot)\backslash G(\cdot)$

Unramified Computation

The Generating Functions

We fix $p \notin S$ and drop p, $\mathbb{Q}_p$ from all notations for this discussion.

$\mathcal{H}$ - spherical Hecke algebra of $G = G\left(\mathbb{Q}_p\right)$ with respect to $K = G\left(\mathbb{Z}_p\right)$.

Existence of Generating Functions

There exists $\Delta_{\chi,s} \in \mathcal{H}\left[\left[p^{-s}\right]\right]$ such that for any $\Lambda \in \pi^*$, $\Re\mathfrak{e}\left(s\right) \gg 0$ and π unramified it holds that $\int\limits_G \Delta_{\chi,s}\left(g\right)\Lambda\left(\pi\left(g\right)v_0\right)dg = \mathcal{L}\left(s,\pi,\chi,\mathfrak{st}\right)\Lambda\left(v_0\right)$.

Goal

$$\int\limits_{U\backslash G} F^*\left(\Psi_E,\chi,g,s\right)\Lambda\left(\pi\left(g\right)v_0\right)dg = \int\limits_{U\backslash G} \Delta^{\Psi_E}_{\chi,s+\frac{1}{2}}\left(g\right)\Lambda\left(\pi\left(g\right)v_0\right)dg$$

for any $\Lambda \in \mathrm{Hom}_U\left(\pi,\Psi_E\right)$. Better yet, prove $F^*\left(\Psi_E,\chi,\cdot,s\right) = \Delta^{\Psi_E}_{\chi,s+\frac{1}{2}}$.

Obstacle

$\Delta_{\chi,s}\left(g\right)$ is complicated.

Approximation to the Generating Function

Let $D_s \in \mathcal{H}[[p^{-s}]]$ $\quad D_s(ktk') = \left|\omega_1(t)\right|^{s+\frac{7}{2}} \qquad \forall t \in T^+,\ k, k' \in K.$

Theorem

$$\exists P_s \in \mathcal{H}[p^{-s}]: \quad D_s = \Delta_{s+\frac{1}{2}} * P_s \quad \forall\, \Re\mathfrak{e}(s) \gg 0.$$

More precisely $P_s = \frac{P_0\left(p^{-s-\frac{1}{2}}\right)A_0 - P_1\left(p^{-s-\frac{1}{2}}\right)A_1}{\zeta_{\mathbb{Q}_p}\left(s+\frac{3}{2}\right)\zeta_{\mathbb{Q}_p}\left(s+\frac{7}{2}\right)\zeta_{\mathbb{Q}_p}\left(s+\frac{1}{2}\right)}$, where $P_0, P_1 \in \mathbb{C}[p^{-s}]$ and $A_0, A_1 \in \mathcal{H}$.
Furthermore, for $\Re\mathfrak{e}(s) \gg 0$ the operator $*P_s$ is injective.

Proof

The first assertion follows from a direct computation using Macdonalds formula. We prove the second assertion. $\mathcal{H}[p^{-s}] \subset L^1(G)$ and hence $\mathcal{H}[p^{-s}] \hookrightarrow \mathcal{B}\left(L^2(G)\right)$. As $\frac{P_1\left(p^{-s-\frac{1}{2}}\right)}{P_0\left(p^{-s-\frac{1}{2}}\right)} \xrightarrow[\Re\mathfrak{e}(s)\to\infty]{} 0$, this proves the claim. □

Theorem

1. $D_s^{\Psi_E} \equiv F^*(\Psi_E, \chi, \cdot, s) * P_s$
2. $D_s^{\Psi_E} = \Delta_{s+\frac{1}{2}}^{\Psi_E} * P_s$

Corollary

$$\int_{U \backslash G} F^*(\Psi_E, \chi, g, s)\, \Lambda(\pi(g) v_0)\, dg = \mathcal{L}\left(s + \frac{1}{2}, \pi, \chi, \mathfrak{st}\right) \Lambda(v_0)$$

Poles of $\mathcal{L}^S(s, \pi, \chi, \mathfrak{st})$ and Functorial Lifts

Poles of $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$

$$\int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi(g)\,\mathcal{E}_E^*(\chi,s,f,g)\,dg = \mathcal{L}^S(s,\pi,\chi,\mathfrak{st})\,d_S(\chi,s,\Psi_E,\varphi_S,f_S)$$

$$\Rightarrow \operatorname{ord}_{s=s_0}\mathcal{L}^S(s,\pi,\chi,\mathfrak{st}) \le \operatorname{ord}_{s=s_0}\mathcal{E}_E^*(\chi,s,f,g)\,.$$

The Poles of $\mathcal{E}_E(\chi,s,f,g)$ (S)

	$s=1/2$	$s=3/2$		$s=5/2$
	$\chi^2=1$	$\chi=1$	$\chi=\chi_E$	$\chi=1$
$E=F\times F\times F$	1	2		1
$E=F\times K$	1	1	1	1
E Galois field extension	1	0	1	1
E non-Galois	1	0		1

Conjecture

For any (χ,s_0,E) such that $\mathcal{E}_E^*(\chi,s,f,g)$ admits a pole of order n_0 at $s_0\in\left\{\frac{1}{2},\frac{3}{2}\right\}$ there exist $\pi\in L^2_{cusp.}(\Psi_E)$ such that $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ admits a pole of order n_0 at $s_0+\frac{1}{2}$.

Shadows of Eisenstein Series

CAP representations

Let $P = M \cdot N \subset G$ be a parabolic subgroup, σ a cuspidal unitary representation of the Levi part M and χ a character of M. A cuspidal representation π of $G(\mathbb{A})$ is called **CAP** with respect to (P,σ,χ) if π is nearly equivalent to a subquotient of $\mathrm{Ind}_P^G \sigma \otimes \chi$.

Theorem (Gan-Gurevich-Jiang, S)

Let E be a Galois cubic étale algebra over F. Let n_E be 2 if $E = F \times F \times F$ and 1 otherwise. The following are equivalent

1. $\mathcal{L}^S(s,\pi,\chi_E,\mathfrak{st})$ admits a pole at $s = 2$ of order n_E.
2. $\Theta_{S_E}(\pi) \neq 0$. In particular π is nearly equivalent to $\Theta_E(1)$, where 1 here is the automorphic trivial representation of $S_E(\mathbb{A})$.
3. π is a **CAP** representation with respect to B supporting the Ψ_E-Fourier coefficient.

The Rallis-Schiffmann Lift

- For a square-integrable irreducible representation σ of $\widetilde{SL_2}(\mathbb{A})$, the theta lift $\theta_{14}(\sigma)$ to SO_7 is irreducible and non-cuspidal.
- If σ is Saito-Kurokawa then the restriction of $\theta_{14}(\sigma)$ to $G_2(\mathbb{A})$ is a non-zero cuspidal representation (not necessarily irreducible).

Theorem (Gurevich-S)

For a cuspidal irreducible representation π of $G_2(\mathbb{A})$ the following are equivalent

1. There exists an automorphic irreducible square integrable representation σ of $\widetilde{SL_2}(\mathbb{A})$ such that π is a weak lift of σ.
2. The partial $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,\mathbf{1},st)$ has a pole at $s=2$.

The pole of $\mathcal{L}^S(s,\pi,\mathbf{1},st)$ is simple unless π is a weak lift of τ_0 (such that the Waldspurger lift of τ_0 to $SO(2,1)$ is trivial) in which case the pole is of order 2.

Thank You!

Theorem ($G(\mathbb{Q})$-orbits in $P_E(\mathbb{Q})\backslash H_E(\mathbb{Q})$)

If $P_E\left(\overline{\mathbb{Q}}\right)\mu G\left(\overline{\mathbb{Q}}\right)$ intersects $H_E(\mathbb{Q})$, the intersection is a $G(\mathbb{Q})$-orbit.

Lemma (Stabilizer of the open orbit)

- For an étale cubic algebra E/F one can choose a representative μ_E of the open orbit in $P_E(\mathbb{Q})\backslash H_E(\mathbb{Q})/G(\mathbb{Q})$ so that $\mathrm{Stab}_G(P\mu_E) = T_{3\alpha+2\beta}\cdot\ker\Psi_E$.
- For any other $G(\mathbb{Q})$-orbit $P_E(\mathbb{Q})\mu$ in $P_E(\mathbb{Q})\backslash H_E(\mathbb{Q})$ the stabilizer $\mathrm{Stab}_G(P\mu)$ contains a unipotent subgroup of $G(\mathbb{Q})$

Corollary

For the non-open $G(\mathbb{Q})$-orbit $P_E(\mathbb{Q})\mu$ it holds that

$$\int_{\mathrm{Stab}_{G(\mathbb{Q})}(\mu)\backslash G(\mathbb{A})} \varphi(g)\, f_s(\mu g)\, dg = 0.$$

Proposition ($G\left(\overline{\mathbb{Q}}\right)$-orbits in $P_E\left(\overline{\mathbb{Q}}\right)\backslash H_E\left(\overline{\mathbb{Q}}\right)$)

There is a bijection between:

- the right GL_2-orbits in $(B_0 \times B_0 \times B_0) \backslash (GL_2 \times GL_2 \times GL_2)$ (where B_0 is the Borel subgroup of GL_2) and
- the right G-orbits of $P_E \backslash H_E$.

Since $B_0 \backslash GL_2 = \mathbb{P}^1$ and the right action of GL_2 is by Möbius transformations, there are 5 such orbits:

1. The open orbit, given by triples $(a, b, c) \in \left(\mathbb{P}^1\right)^3$, $a \neq b \neq c$.
2. Orbits of the form $(a, a, b) \in \left(\mathbb{P}^1\right)^3$, $(a, b, a) \in \left(\mathbb{P}^1\right)^3$ or $(a, b, b) \in \left(\mathbb{P}^1\right)^3$, $a \neq b \neq c$.
3. The orbit of $(a, a, a) \in \left(\mathbb{P}^1\right)^3$