

# Poles of the Standard $\mathcal{L}$ -function and Functorial Lifts for $G_2$

Avner Segal

Department of Mathematics  
University of British Columbia  
avners@math.ubc.ca

1 A Crash Course in Functoriality

2 Realizations of Functoriality

3 Functoriality and  $G_2$

# $p$ -adic Fields and Ring of Adeles

- For each prime  $p$  we let  $\mathbb{Q}_p$  denote the completion of  $\mathbb{Q}$  with respect to the absolute value

$$\left| \frac{m}{n} p^k \right|_p = p^{-k}.$$

This is an ultrametric field and hence locally compact and totally disconnected.

- $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$  is the ring of  $p$ -adic integers. It is a local ring and  $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$ .
- Notation:  $\mathbb{Q}_\infty = \mathbb{R}$ .
- $\mathbb{A} = \prod'_{p \leq \infty} \mathbb{Q}_p = \{(x_p) \in \prod_{p \leq \infty} \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ for almost all } p\}$ .
- $\mathbb{A}$  is a locally-compact normed ring.
- $\mathbb{Q} \hookrightarrow \mathbb{A}$  is a discrete subring and the quotient  $\mathbb{Q} \backslash \mathbb{A}$  is compact.
- $\mathbb{Q}^\times \backslash \mathbb{A}^1$  has finite measure (but not compact).

# Square-Integrable Automorphic Representations

- Throughout this talk,  $G$  is a split, simple and connected algebraic group defined over  $\mathbb{Z}$  (e.g.  $SL_n$ ,  $PGL_n$ ,  $Sp_{2n}$ ,  $SO_m$ ,  $G_2$ , etc.).
- Consider the right-regular (complex) representation of  $G(\mathbb{A})$ :

$$L^2(G(\mathbb{Q}) \backslash G(\mathbb{A})) = L^2_{disc.}(G) \oplus L^2_{cont.}(G)$$

## Example - Groups with Discrete and Continuous Spectrum

- Discrete spectrum:  $L^2(S^1) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} \cdot e^{2\pi i x n}$ .
- Continuous spectrum:  $L^2(\mathbb{Z}) = \int_{S^1}^{\oplus} (\mathbb{C} \cdot e^{2\pi i x n}) dx$ .
- A square-integrable automorphic representation is a unitary subrepresentation  $\pi \subset L^2_{disc.}(G)$ . Automorphic representations are **embedded** representations!

- We say that a form  $\varphi \in L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$  is **cuspidal** if

$$\int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} \varphi(ug) du = 0$$

for any unipotent radical  $U$  of any proper parabolic subgroup of  $G$  and  $g \in G(\mathbb{A})$ .

- The space of cuspidal forms is denoted by  $L^2_{\text{cusp.}} \subset L^2_{\text{disc.}}(G)$ .

$$L^2(G(\mathbb{Q}) \backslash G(\mathbb{A})) = L^2_{\text{disc.}} \oplus L^2_{\text{cont.}} = L^2_{\text{cusp.}} \oplus \underbrace{L^2_{\text{res.}} \oplus L^2_{\text{cont.}}}_{\text{Can be constructed using Eisenstein series}}$$

Can be constructed  
using Eisenstein series

- A **cuspidal representation** is a unitary subrepresentation of  $L^2_{\text{cusp.}}$ . Cuspidal representations are **embedded** representations!

# Irreducible Representations of $G(\mathbb{A})$

## Theorem (Flath)

For an irreducible admissible representation  $\pi$  of  $G(\mathbb{A})$  we have  $\pi = \otimes'_{p \leq \infty} \pi_p$ , where  $\pi_p$  is unramified for almost all  $p$ .

- $\pi_p : G(\mathbb{Q}_p) \rightarrow \text{End}(V_{\pi_p})$  is said to be **unramified** if it contains a non-zero  $G(\mathbb{Z}_p)$ -fixed vector.

$$\otimes'_{p \leq \infty} \pi_p = \text{Span}_{\mathbb{C}} \left\{ \otimes_{p \leq \infty} v_p \mid v_p = v_p^0 \text{ for almost all } p \right\}.$$

- $\pi = \otimes'_{p \leq \infty} \pi_p$  and  $\pi' = \otimes'_{p \leq \infty} \pi'_p$  are said to be **nearly-equivalent** if  $\pi_p \cong \pi'_p$  for almost all  $p$ .
- If  $\pi_p \cong \pi'_p$  for all  $p$ , then  $\pi \cong \pi'$ . However, for automorphic representations, it is possible that  $\pi \neq \pi'$ .

# Dual Groups

- For a split, simple and connected algebraic group  $G$ , Langlands defined a complex Lie group  $G^\vee$  called the **complex dual group of  $G$** .
- Examples:

$$G = SL_n \rightsquigarrow G^\vee = PGL_n(\mathbb{C}), \quad G = PGL_n \rightsquigarrow G^\vee = SL_n(\mathbb{C}), \quad (A_n)$$

$$G = Sp_{2n} \rightsquigarrow G^\vee = SO_{2n+1}(\mathbb{C}), \quad G = SO_{2n+1} \rightsquigarrow G^\vee = Sp_{2n}(\mathbb{C}), \quad (BC_n)$$

$$G = SO_{2n} \rightsquigarrow G^\vee = SO_{2n}(\mathbb{C}), \quad (D_n)$$

$$G = G_2 \rightsquigarrow G^\vee = G_2(\mathbb{C}). \quad (G_2)$$

# Satake Parameters

## Theorem (Satake, Langlands)

There is a bijection

$$\left\{ \begin{array}{c} \text{Equivalence classes} \\ \text{of unramified} \\ \text{representations of } G(\mathbb{Q}_p) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Conjugacy classes} \\ \text{of semi-simple} \\ \text{elements in } G^\vee \end{array} \right\} .$$

$$\pi_p \qquad \longleftrightarrow \qquad t_{\pi_p}$$

## Example - $G = SL_2$

Let  $G = SL_2$ , and let  $\chi : B \rightarrow \mathbb{C}^\times$  be given by  $\chi \left( \begin{smallmatrix} t & x \\ & t^{-1} \end{smallmatrix} \right) = |t|^s$ . The

parabolic induction  $\text{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi$  admits a unique unramified constituent

$\pi_{p,s}$ . The Satake parameter of  $\pi_{p,s}$  is the conjugacy class of

$$t_{\pi_{p,s}} = \begin{pmatrix} p^{-s} & \\ & 1 \end{pmatrix} \text{ in } PGL_2(\mathbb{C}).$$



## Functoriality Principle

Given  $G, H, r : H^\vee \rightarrow G^\vee$ , one expects a map

$$\begin{aligned} L_{disc.}^2(H) &\xrightarrow{r_*} L_{disc.}^2(G) \\ \tau = \otimes' \tau_p &\longrightarrow \pi = \otimes' \pi_p \end{aligned}$$

such that  $t_{\pi_p} = r(t_{\tau_p})$  for almost every  $p$ . Such  $\pi$  is called a **weak functorial lift** of  $\tau$ .

This gives rise to a map:

$$r_* : \left\{ \begin{array}{l} \text{nearly equivalence classes} \\ \text{of irreducible automorphic} \\ \text{representations of } H(\mathbb{A}) \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{nearly equivalence classes} \\ \text{of irreducible automorphic} \\ \text{representations of } G(\mathbb{A}) \end{array} \right\}$$

$$r_* : \left\{ \begin{array}{l} \text{nearly equivalence classes} \\ \text{of irreducible automorphic} \\ \text{representations of } H(\mathbb{A}) \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{nearly equivalence classes} \\ \text{of irreducible automorphic} \\ \text{representations of } G(\mathbb{A}) \end{array} \right\}$$

## Question

Given an irreducible automorphic representation  $\pi = \otimes \pi_p$  of  $G(\mathbb{A})$ , is  $\pi$  in the image of  $r_*$ ?

## Possible Approach

Use an explicit construction

$$\mathcal{B} : L_{disc.}^2(G) \rightarrow L_{disc.}^2(H)$$

such that if  $\mathcal{B}(\pi) \neq 0$  then for any irreducible representation  $\tau \subset \mathcal{B}(\pi)$  it holds that  $\pi = r_*(\tau)$ . Such  $\tau$  is called a **backward lift** of  $\pi$ .

## Example - $\theta$ -lift

- Dual reductive pair: Assume that  $G \times H \subset M$  so that  $\text{Cent}_M(G) = H$  and  $\text{Cent}_M(H) = G$ .
- Example:  $G = \widetilde{\text{Sp}}_{2n}$  and  $H = \text{SO}_m$  (or vice versa).
- Assume that  $M(\mathbb{A})$  admits a minimal representation  $\Pi \subset L_{\text{res.}}^2(M)$ .
- For  $\pi \subset L_{\text{cusp.}}^2(G)$  irreducible, consider the representation  $\Theta(\pi)$  of  $H(\mathbb{A})$  spanned by:

$$\Theta(\varphi)(h) = \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi(g) \theta(g, h) dg, \quad \varphi \in \pi, \theta \in \Pi.$$

## Question

Is  $\Theta(\pi) \neq 0$ ? If so, is  $\pi$  a weak functorial lift?

- For  $\rho : G^{\vee} \rightarrow GL_N(\mathbb{C})$  and a finite set of primes  $S \ni \infty$  such that  $\pi$  is unramified outside of  $S$ , Langlands constructed the partial  $\mathcal{L}$ -function:

$$\mathcal{L}^S(\pi, s, \rho) = \prod_{p \notin S} \det(1 - \rho(t_{\pi_p}) p^{-s})^{-1}.$$

- For  $G = GL_1$ ,  $\pi = \mathbf{1}$ ,  $S = \{\infty\}$  and  $\rho = \mathbf{1}$ :  $\mathcal{L}^S(\pi, s, \rho) = \zeta(s)$
- This product converges for  $\Re(s) \gg 0$ .

### Conjecture (Langlands)

$\mathcal{L}^S(\pi, s, \rho)$  admits a meromorphic continuation to  $\mathbb{C}$ .

$$H^{\vee} \xrightarrow{r_*} G^{\vee} \xrightarrow{\rho} GL_N(\mathbb{C})$$

### Baby Example - $\mathcal{L}$ -functions and Weak Lifts

If  $\rho$  is chosen so that  $\rho \circ r = \rho' \oplus \mathbf{1}$  then

$$\mathcal{L}_G^S(r_*(\tau), s, \rho) = \mathcal{L}_H^S(\tau, s, \rho \circ r) = \zeta^S(s) \mathcal{L}_H^S(\tau, s, \rho'). \quad (*)$$

If  $\mathcal{L}_H^S(\tau, s, \rho')$  is holomorphic and non-vanishing at  $s = 1$  then  $\mathcal{L}_G^S(r_*(\tau), s, \rho)$  has a simple pole at  $s = 1$ . Usually, one sees multiple (shifted)  $\zeta$ -functions and Dirichlet  $\mathcal{L}$ -functions.

### Goal - Proving the Converse (Prototype)

If  $\mathcal{L}^S(\pi, s, \rho)$  has a pole at  $s = s_0$  of order  $n$  then  $\pi = r_*(\tau)$  for some automorphic representation  $\tau$  of  $H(\mathbb{A})$ .

## Goal - Proving the Converse (Prototype)

If  $\mathcal{L}^S(\pi, s, \rho)$  has a pole at  $s = s_0$  of order  $n$  then  $\pi = r_*(\tau)$  for some automorphic representation  $\tau$  of  $H(\mathbb{A})$ .

## Main Tool - Integral Representations of $\mathcal{L}$ -functions

$$\mathcal{Z}(s, \varphi, \dots) = \mathcal{L}^S(\pi, s, \rho) \cdot \boxed{\text{ramified factor}},$$

## Example

$$\mathcal{Z}(s, \varphi, \mathcal{E}) = \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi(g) \mathcal{E}(g, s) dg,$$

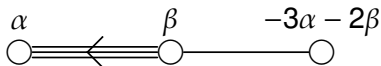
where  $\varphi \in \pi$  and  $\mathcal{E}$  is an Eisenstein series on a larger group  $G'$ .

# General Strategy (Rallis)

- Assume that  $\mathcal{L}^S(\pi, s, \rho)$  admits a pole of order  $n$  at  $s = s_0$ .
- Then  $(s - s_0)^n \mathcal{Z}(\varphi, s, \dots) \neq 0$  on  $\pi$ .
- Show that  $\mathcal{B}(\pi) \neq 0$  (these requires that the construction of  $\mathcal{Z}(\varphi, s, \dots)$  will encode  $\mathcal{B}(\pi)$ ).
- Let  $\tau \subseteq \mathcal{B}(\pi)$  be an irreducible automorphic representation. Show that  $\pi = r_*(\tau)$ .
- Kudla-Rallis implemented this strategy for  $(\widetilde{Sp}_{2n}, SO_m)$ .
- Today: Implementation for  $G = G_2$  and various  $H$  and  $r$ .

# The Group $G_2$

- Let  $\mathbb{O}$  denote a (split) Cayley algebra of dimension 8 over  $\mathbb{Q}$  named the **octonions**. This is a **non-associative** composition algebra.
- The group of automorphisms of  $\mathbb{O}$  is a split, simple, adjoint and simply-connected exceptional group of type  $G_2$ .
- There is a natural map  $Nm : \mathbb{O} \rightarrow \mathbb{Q}$  so  $G_2$  admits a natural embedding to  $SO_8$ .
- In fact,  $G_2$  preserves the decomposition  $\mathbb{O} = \mathbb{Q} \oplus \mathbb{O}^0$ , where  $\mathbb{O}^0$  are the imaginary octonions and hence  $G_2$  admits an embedding  $st$  into  $SO_7$ .



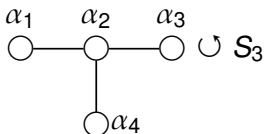
$$r_1 : SL_3(\mathbb{C}) \rightarrow G_2(\mathbb{C})$$

$$r_2 : SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \rightarrow G_2(\mathbb{C})$$



$$\left\{ \begin{array}{l} \text{Iso. classes} \\ \text{of étale cubic} \\ \text{algebras over } \mathbb{Q} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Iso. classes of} \\ \text{Quasi-split} \\ \text{forms of } D_4/\mathbb{Q} \end{array} \right\}$$

$$E \longleftrightarrow H_E = \text{Spin}_8^E$$



- $(\text{Spin}_8)^{S_3} = G_2$ .
- In fact,  $H_E^{S_E} = G_2$ , where  $S_E = \text{Aut}_{\mathbb{Q}}(E)$ .
- $G_2 \times S_E \subset H_E \rtimes S_E$  is a dual reductive pair.

# A Rankin-Selberg Integral

- Let  $\pi \boxtimes \chi$  be a cuspidal representation of  $G_2(\mathbb{A}) \times GL_1(\mathbb{A})$  and let  $\mathcal{E}(\chi, f, s, g)$  be a (normalized) degenerate Eisenstein series on  $Spin_8^E(\mathbb{A})$  (induced from the Heisenberg parabolic subgroup).

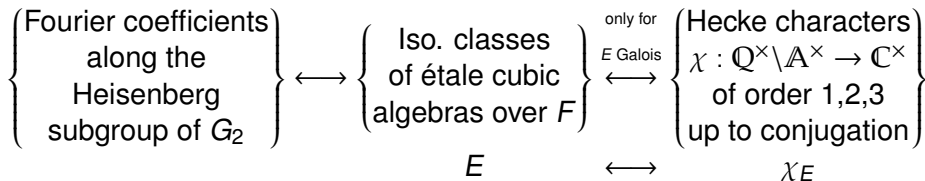
## Theorem (S)

For  $\varphi \in \pi$ :

$$\begin{aligned} \mathcal{Z}(\chi, \varphi, f, s) &= \int_{G_2(\mathbb{Q}) \backslash G_2(\mathbb{A})} \varphi(g) \mathcal{E}\left(\chi, f, s - \frac{1}{2}, g\right) dg \\ &= \mathcal{L}^S(s, \pi \boxtimes \chi, st) d_S(f_S, \varphi_S, s). \end{aligned}$$

For any  $s_0$  there exist data so that  $d_S(f_S, \varphi_S, s)$  is holomorphic and non-zero.

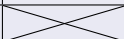
Both  $\mathcal{L}^S(s, \pi \boxtimes \chi, st)$  and  $d_S(f_S, \varphi_S, s)$  admits meromorphic continuation to  $\mathbb{C}$ .  $d_S(f_S, \varphi_S, s)$  is non-zero when  $E$  is matched with a non-zero Fourier-coefficient of  $\pi$ .



### Corollary

$$\text{ord}_{s=s_0} \mathcal{L}^S(s, \pi, \chi, st) \leq \text{ord}_{s=s_0} \mathcal{E}\left(\chi, s - \frac{1}{2}, f, g\right).$$

### The Poles of $\mathcal{E}_E\left(\chi, s - \frac{1}{2}, f, g\right)$ for $\Re(s) > 0$ (S)

	$s = 1$	$s = 2$		$s = 3$
	$\chi^2 = 1$	$\chi = 1$	$\chi = \chi_E$	$\chi = 1$
$E = F \times F \times F$	1	2		1
$E = F \times K$	1	1	1	1
$E$ Galois field extension	1	0	1	1
$E$ non-Galois	1	0		1

# Backward Lift for $r_{1*}$ : The Gan-Gurevich-Jiang Lift

- (Gan, Gurevich, Jiang) The residual representation of  $\mathcal{E}(\chi_E, f, s - \frac{1}{2}, g)$  at  $s = 2$  is the minimal representation  $\Pi_E$ .
- The dual reductive pair  $G_2 \times S_E$  in  $Spin_8^E \times S_E$  gives rise to an exceptional  $\Theta$ -lift.
- Assume that  $\mathcal{L}^S(s, \pi, \chi_E, st)$  has a pole of maximal order at  $s = 2$ .

$$\begin{aligned}
 0 &\neq \text{Res}_{s=2} \left[ \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi(g) \mathcal{E}(\chi, f, s, g) dg \right] \\
 &= \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi(g) \text{Res}_{s=2} [\mathcal{E}(\chi, f, s, g)] dg \\
 &= \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi(g) \theta(g) dg = \Theta_{S_E}(\varphi)(1)
 \end{aligned}$$

## Theorem (S)

IF  $\mathcal{L}^S(s, \pi, \chi_E, st)$  admits a pole of maximal order at  $s = 2$  then  $\Theta_{S_E}(\pi) \neq 0$ .

# So what did that had to do with $r_{1*}$ ?

General Theory	GGJ-lift
$G$	$GL_1 \times G_2$
$G^\vee$	$\mathbb{C}^\times \times G_2(\mathbb{C})$
$H$	$PGL_3, PD^\times$
$H^\vee$	$SL_3(\mathbb{C})$
$r : H^\vee \rightarrow G^\vee$	$r_1 : SL_3(\mathbb{C}) \rightarrow G_2(\mathbb{C})$
$\rho : G^\vee \rightarrow GL_N(\mathbb{C})$	$\rho = \text{st} : \mathbb{C}^\times \times G_2(\mathbb{C}) \rightarrow GL_7(\mathbb{C})$
“ $\text{ord}_{s=s_0} \mathcal{L}^S(s, \pi, \rho) = n$ $\Rightarrow \exists \tau : r_{1*}(\tau) = \pi$ ”	$\text{ord}_{s=2} \mathcal{L}^S(s, \pi, \chi_E, \text{st}) = n_E$ $\Rightarrow \Theta_{S_E}(\pi) \neq 0$ $\stackrel{\text{Gan}}{\Rightarrow} r_{1*}(\chi_E) = \pi, \quad \text{when } \chi_E^3 = \mathbf{1}$

## Remark

All automorphic representations of  $S_E(\mathbb{A})$  are nearly equivalent to the trivial representation.

# Backward Lift for $r_{2^*}$ : The Rallis-Schiffmann Lift

- For a square-integrable irreducible representation  $\sigma$  of  $\widetilde{SL}_2(\mathbb{A})$ , the theta lift  $\theta_{14}(\sigma)$  to  $SO_7$  is irreducible and non-cuspidal.
- If  $\sigma$  is Saito-Kurokawa ( $\theta_{10}(\sigma) \neq 0$ ) then the restriction of  $\theta_{14}(\sigma)$  to  $G_2(\mathbb{A})$  is a non-zero cuspidal representation (not necessarily irreducible).

## Theorem (Gurevich-S)

For a cuspidal irreducible representation  $\pi$  of  $G_2(\mathbb{A})$  the following is equivalent

- 1 There exists an automorphic irreducible square integrable representation  $\sigma$  of  $\widetilde{SL}_2(\mathbb{A})$  such that  $\pi$  is a weak lift of  $\sigma$ .
- 2 The partial  $\mathcal{L}$ -function  $\mathcal{L}^S(s, \pi, \mathbf{1}, st)$  has a pole at  $s = 2$ .

The pole of  $\mathcal{L}^S(s, \pi, \mathbf{1}, st)$  is simple unless  $\Theta_{S_3}(\pi) \neq 0$

- Assume that  $\mathcal{L}^S(s, \pi, \mathbf{1}, st)$  has a simple pole at  $s = 2$ .
- It follows that  $\pi$  supports an  $F \times K$ -Fourier coefficient, where  $K \neq F \times F$ .
- It follows that there exist  $\varphi \in \pi$  and  $\mathcal{E}(\mathbf{1}, s, f, g)$

$$\begin{aligned}
 0 &\neq \operatorname{Res}_{s=2} \left[ \int_{G_2(\mathbb{Q}) \backslash G_2(\mathbb{A})} \varphi(g) \mathcal{E}(\mathbf{1}, s, f, g) dg \right] \\
 &= \int_{G_2(\mathbb{Q}) \backslash G_2(\mathbb{A})} \varphi(g) \operatorname{Res}_{s=2} [\mathcal{E}(\mathbf{1}, s, f, g)] dg,
 \end{aligned}$$

### Theorem - Siegel-Weil Type Identity

$$\operatorname{Res}_{s=2} [\mathcal{E}_{F \times K}(\mathbf{1}, s, f, g)] = \tilde{\mathcal{E}}(s, \tilde{f}, g) \Big|_{s=2},$$

where  $\tilde{\mathcal{E}}$  is a degenerate Eisenstein series associated to the Siegel parabolic subgroup  $Q$  of  $H_{F \times K}$ .

- Let  $\Pi_Q$  denote the residual representation of  $\widetilde{\mathcal{E}}$ .
- It follows that there exist  $\eta \in \Pi_Q$  such that

$$\begin{aligned} 0 &\neq \int_{G_2(\mathbb{Q}) \backslash G_2(\mathbb{A})} \varphi(g) \eta(g) dg \\ &= \int_{G_2(\mathbb{Q}) \backslash G_2(\mathbb{A})} \varphi(g) \bar{\eta}(g) dg, \end{aligned}$$

where  $\bar{\eta}$  is in the residual representation  $\Pi_{\bar{Q}}$  of a degenerate Eisenstein series in  $SO(V_K^8)$  - the orthogonal group of an 8-dimensional quadratic space with discriminant associated to  $K$ .

- Note that  $V_K^8 = V_{split}^7 \oplus V_K^1$  and that  $G_2 \subset SO(V_{split}^7)$ .



- Consider the following see-saw diagram in  $\widetilde{Sp}_{16}$

$$\begin{array}{ccccc}
 RS_\psi(\varphi) \times \theta(\mathbf{1}) & \widetilde{SL}_2 \times \widetilde{SL}_2 & SO(V_K^8) & \bar{\eta} = \theta(\mathcal{E}(\chi_K, \underline{f}, \underline{s})) & \\
 & \uparrow & \uparrow & & \\
 & \Delta & & & \\
 \mathcal{E}(\chi_K, \underline{f}, \underline{s}) & \widetilde{SL}_2 & G_2 \times SO(V_K^1) & \varphi \times \mathbf{1} & \\
 & \downarrow & \downarrow & & 
 \end{array}$$

$$\begin{aligned}
 & \int_{G_2(\mathbb{Q}) \backslash G_2(\mathbb{A})} \varphi(g) \theta(\mathcal{E}(\chi_K, \underline{f}, \underline{s}))(g) dg \neq 0 \\
 \Rightarrow & \int_{\widetilde{SL}_2(\mathbb{Q}) \backslash \widetilde{SL}_2(\mathbb{A})} RS_\psi(\varphi)(h) \theta(\mathbf{1})(h) \mathcal{E}(\chi_K, \underline{f}, \underline{s}, h) dh \neq 0
 \end{aligned}$$

- We conclude that  $RS_\psi(\varphi) \neq 0$ .

# So what did that had to do with $r_{2^*}$ ?

General Theory	RS-lift
$G$	$G_2$
$G^\vee$	$G_2(\mathbb{C})$
$H$	$PGL_2 \times PGL_2$
$H^\vee$	$SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$
$r : H^\vee \rightarrow G^\vee$	$r_2 : SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \rightarrow G_2(\mathbb{C})$
$\rho : G^\vee \rightarrow GL_N(\mathbb{C})$	$\rho = st : G_2(\mathbb{C}) \rightarrow GL_7(\mathbb{C})$
$\text{“ord}_{s=s_0} \mathcal{L}^S(s, \pi, \rho) = n$ $\Rightarrow \exists \tau : r_{1^*}(\tau) = \pi$	$\text{ord}_{s=2} \mathcal{L}^S(s, \pi, st) = 1$ $\Rightarrow RS_\psi(\pi) \neq 0$ $\xRightarrow{RS} r_{2^*}(\tau \boxtimes \mathbf{1}) = \pi, \quad \tau = Wd(RS_\psi(\pi))$

# Thank You!