$\mathcal{L}$-functions and the Failure of the Local-Global Principle for Automorphic Representations of G_2

Avner Segal

Department of Mathematics
University of British Columbia
avners@math.ubc.ca

- Given a finite dimensional space V over a field F ($char\,(F) \neq 2$), a non-degenerate quadratic form on V is a map $q : V \to F$ such that
  - $q\,(\alpha v) = \alpha^2 q\,(v)$ for any $\alpha \in F$ and $v \in V$.
  - $b\,(u, v) = q\,(u + v) - q\,(u) - q\,(v)$ is a non-degenerate symmetric bilinear form.
- $V = F^n$: $q\,(x_1, ..., x_n) = \sum_{i,j=1}^{n} a_{i,j} x_i x_j$ (can take $a_{i,j} = a_{j,i}$).
- We say that q represents 0 if there there exists $v \in V \setminus \{0\}$ such that $q\,(v) = 0$.

Examples - Binary cubic forms

Take $q_1\,(x, y) = x^2 + y^2$ and $q_2\,(x, y) = x^2 - y^2$. The form q_2 represents 0 over $\mathbb{Q}$ while q_1 does not. $q_3\,(x, y) = x^2 - 2y^2$ also doesn't represent 0.

Hasse-Minkowski Theorem, 1924

Given a quadratic form q over $\mathbb{Q}$, q represents 0 if and only if q represents 0 over all $\mathbb{Q}_p$ (including $\mathbb{Q}_\infty = \mathbb{R}$).

p-adic Numbers

Ostrowski, 1916

Up to equivalence, any non-trivial absolute value on $\mathbb{Q}$ is either the standard absolute value or a p-adic absolute value for a prime p, namely $\left|\frac{m}{n}p^k\right|_p = p^{-k}$.

- For each prime p we let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|_p$;this is an ultrametric field and hence locally compact and totally disconnected.
- $\mathbb{Z}_p = \left\{x \in \mathbb{Q}_p \,\middle|\, |x|_p \leq 1\right\}$ is the ring of p-adic integers. It is a local ring and $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$.
- For a quadratic space (q, V) over $\mathbb{Q}$, we denote by q_p the induced quadratic form on $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$.
- $q_3(x, y) = x^2 - 2y^2$ doesn't represent 0 over $\mathbb{Q}_2$ because $\left|x^2\right|_2 \neq \left|2y^2\right|_2$ for $(x, y) \neq 0$.
- Notation: $\mathbb{Q}_\infty = \mathbb{R}$.

Hasse-Minkowski Theorem, 1924

Given a quadratic form q over $\mathbb{Q}$, q represents 0 if and only if q represents 0 over all $\mathbb{Q}_p$ (including $\mathbb{Q}_\infty = \mathbb{R}$).

Remark

The Hasse-Minkowski theorem cannot be generalized to forms of higher degree.

Question

Given non-degenerate quadratic forms q_p of dimension n for all $p \leq \infty$, is there a quadratic form $\widetilde{q}$ over $\mathbb{Q}$ such that $q_p = \widetilde{q}_p$ for all p?

- What invariants determine a (non-deg.) quadratic form over $\mathbb{Q}_p$?
  - $\mathbb{Q}_\infty = \mathbb{R}$: dimension n and signature (r, s).
  - $\mathbb{Q}_p$ ($p < \infty$): dimension n, discriminant $d_p \in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ and Hasse invariant $\epsilon_p \in \{\pm 1\}$.

Corollary (to Hasse-Minkowski)

- For a quadratic form q, the Hasse invariant ϵ_p of q_p is 1 for almost all p and the product of all Hasse invariants ϵ_p is 1.
- The converse is also true. Choosing $n \in \mathbb{N}$, $d \in \mathbb{Q}^\times / (\mathbb{Q}^\times)^2$ and $\epsilon_p \in \{\pm 1\}$ for all p assume that:
  - Only finitely many of them are -1.
  - $\prod_p \epsilon_p = 1$.
  - $\epsilon_\infty = (-1)^{\frac{s(s-1)}{2}}$, $sign(d) = (-1)^s$.

  Then there exists a (unique up to equivalence) non-degenerate quadratic space (V, q) over $\mathbb{Q}$ of dimension n with local invariants $\left(\epsilon_p, d\left(\mathbb{Q}_p^\times\right)^2\right)$.

Question

Given $d_p \in \mathbb{Q}_p^\times / \left(\mathbb{Q}_p^\times\right)^2$ for every p, is there $d \in \mathbb{Q}^\times / (\mathbb{Q}^\times)^2$ such that $d\left(\mathbb{Q}_p^\times\right)^2 = d_p\left(\mathbb{Q}_p^\times\right)^2$ for all p?

Dirichlet $\mathcal{L}$-functions

- In his seminal paper from 1837, Dirichlet introduced the following $\mathcal{L}$-series for a Dirichlet character $\chi : \mathbb{Z} \to \mathbb{C}$ (i.e. a function on $\mathbb{Z}$ coming from $\chi : (\mathbb{Z}/m\mathbb{Z})^{\times} \to S^1$):

$$\mathcal{L}(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re\mathfrak{e}(s) > 1.$$

- For $\Re\mathfrak{e}(s) > 1$ one can write $\mathcal{L}(\chi, s)$ as a product

$$\mathcal{L}(\chi, s) = \prod_{p<\infty} \mathcal{L}_p(\chi, s), \quad \mathcal{L}_p(\chi, s) = \frac{1}{1 - \chi(p) p^{-s}}.$$

- Can define $\mathcal{L}_\infty(\chi, s)$ and $L(\chi, s) = \mathcal{L}_\infty(\chi, s) \cdot \mathcal{L}(\chi, s)$ so that

$$L(\overline{\chi}, 1 - s) = \epsilon(\chi, s) \cdot L(\chi, s).$$

Dirichlet $\mathcal{L}$-functions and Discriminants

Question

Given $d_p \in \mathbb{Q}_p^\times / \left(\mathbb{Q}_p^\times\right)^2$ for every p, is there $d \in \mathbb{Q}^\times / \left(\mathbb{Q}^\times\right)^2$ such that $d\left(\mathbb{Q}_p^\times\right)^2 = d_p \left(\mathbb{Q}_p^\times\right)^2$ for all p?

- For any $p < \infty$:

$$d_p \overset{\textbf{LCFT}}{\rightsquigarrow} \chi_p : \mathbb{Q}_p^\times \to \{\pm 1\} \rightsquigarrow \mathcal{L}_p\left(\chi_p, s\right) = \frac{1}{1 - \chi_p\left(p\right) p^{-s}}.$$

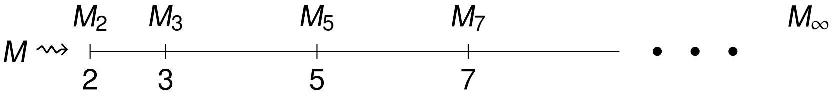
- Can define χ_∞ and $\mathcal{L}_\infty\left(\chi_\infty, s\right)$.
- Consider the product $\prod_{p \le \infty} \mathcal{L}_p\left(\chi_p, s\right)$, the product converges for $\Re\mathfrak{e}\left(s\right) \gg 0$.
- If the product admits a meromorphic continuation and a functional equation then it equals $\mathcal{L}\left(\chi, s\right)$ for a quadratic Dirichlet character χ (by Kaczorowski-Perelli converse theorem).
- $\chi \overset{\textbf{GCFT}}{\rightsquigarrow} d \in \mathbb{Q}^\times \backslash \left(\mathbb{Q}^\times\right)^2$.

Question I (Existence)

When does an (arbitrary) collection $\left\{M_p\right\}$ of "local" objects arise from a "global" object M? What are the conditions on $\left\{M_p\right\}$ so it will happen?

Question II (Uniqueness)

Is a "global" object M uniquely determined by its "local" constituents M_p? If not, can we characterize all the "global" objects M' with "local" constituents $\left\{M_p\right\}$?



The Ring of Adeles

- $\mathbb{A} = \left\{ \left(x_p \right) \in \prod_{p \leq \infty} \mathbb{Q}_p \,\middle|\, x_p \in \mathbb{Z}_p \text{ for almost all } p \right\}$.
  This construction is called a **restricted product.**
- Another description: $\mathbb{A} = \varinjlim_{S} \left(\prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p \right)$.
- Yet another way: $\mathbb{A} = \left(\mathbb{Q} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \right) \times \mathbb{R}$.
- $\mathbb{A}$ is a locally-compact normed ring.
- $\mathbb{Q} \hookrightarrow \mathbb{A}$ is a discrete subring and the quotient $\mathbb{Q} \backslash \mathbb{A}$ is compact.
- $\mathbb{Q}^{\times} \backslash \mathbb{A}^1$ has finite measure (but is not compact).
- For all m: $(\mathbb{Z}/m\mathbb{Z})^{\times} \cong \mathbb{Q}^{\times} \backslash \mathbb{A}^{\times} / K_m$. This gives rise to a bijection:

$$\{\text{Dirihclet characters}\} \longleftrightarrow \left\{ \chi : \mathbb{Q}^{\times} \backslash \mathbb{A}^{\times} \to S^1 \right\}$$

Adelic groups

- Let G be a nice (split & simple) algebraic group (scheme) over $\mathbb{Z}$, e.g. SL_n, symplectic groups, orthogonal groups, exceptional groups...
- We wish to consider $G(\mathbb{A})$, the $\mathbb{A}$-points of G.
- For any $p < \infty$, $G\left(\mathbb{Z}_p\right)$ is a maximal compact subgroup of $G\left(\mathbb{Q}_p\right)$.
- $G(\mathbb{A}) = \left\{\left(g_p\right) \in \prod_{p\leq\infty} G\left(\mathbb{Q}_p\right) \,\middle|\, g_p \in G\left(\mathbb{Z}_p\right) \text{ for almost all } p\right\}$.
- $G(\mathbb{A})$ is locally-compact, $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{A})$ and $G(\mathbb{Q})\backslash G(\mathbb{A})$ has finite volume.

Cuspidal Representations

- A classical question in number theory is to decompose $L^2(\Gamma\backslash SL_2(\mathbb{R}))$, where Γ is a discrete subgroup of $SL_2(\mathbb{R})$.
- Hidden symmetries yield an embedding of $L^2(\Gamma\backslash SL_2(\mathbb{R}))$ in $L^2(SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}))$.
- We wish to study $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ as a representation of $G(\mathbb{A})$.
- We say that a form $\varphi \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is **cuspidal** if it holds that

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\,du = 0$$

for any unipotent radical U of any proper parabolic subgroup of G and almost all $g \in G(\mathbb{A})$.

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = L^2_{disc.} \oplus L^2_{cont.} = L^2_{cusp.} \oplus \underbrace{L^2_{res.} \oplus L^2_{cont.}}_{\text{Can be constructed using Eisenstein series}}$$

- A subrepresentation $\pi \subset L^2_{cusp.}$ is called **Cuspidal.**
- Cuspidal representations are **embedded** representations!

Structure of Irreducible Representations of $G(\mathbb{A})$

Theorem (Flath, 1979)

Any admissible irreducible (abstract) representation π of $G(\mathbb{A})$ is the restricted tensor product $\otimes'_{p\leq\infty}\pi_p$ of irreducible representations π_p of $G(\mathbb{Q}_p)$ such that π_p is unramified for almost all p.

- A representation π_p of $G(\mathbb{Q}_p)$ is said to be **unramified** if there exists a non-zero vector fixed by $G(\mathbb{Z}_p)$, i.e.

$$0 \neq v_p^0 \in \pi_p^{G(\mathbb{Z}_p)}.$$

- The restricted tensor product is defined to be:

$$\otimes'_{p\leq\infty}\pi_p = Span_{\mathbb{C}}\left\{\otimes_{p\leq\infty} v_p \;\middle|\; v_p = v_p^0 \text{ for almost all } p\right\}.$$

Automorphic $\mathcal{L}$-functions

Theorem (Satake, Langlands)

$$\left\{\begin{matrix}\text{equivalence classes}\\ \text{of unramified}\\ \text{representations of } G\left(\mathbb{Q}_p\right)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{conjugacy classes}\\ \text{of semi-simple}\\ \text{elements in } {}^LG\end{matrix}\right\}$$

$$\pi_p \longleftrightarrow t_{\pi_p}$$

- For $\pi = \otimes'_{p\leq\infty}\pi_p$, ρ a finite dimensional representation of ${}^LG$, a finite set of primes $S \ni \infty$ and $s \in \mathbb{C}$ we let

$$\mathcal{L}^S(s, \pi, \rho) = \prod_{p \notin S} \frac{1}{det\left(1 - \rho\left(t_{\pi_p}\right)p^{-s}\right)}.$$

Local-Global Principle for GL_n

Question II (Uniqueness)

Is a cuspidal representation π of $GL_n(\mathbb{A})$ uniquely determined by its local constituents π_p? YES

Multiplicity One Theorem (Jacquet, Langlands, Piatetski-Shapiro, Shalika)

For any irreducible representation π of $GL_n(\mathbb{A})$, the multiplicity of π in the space $L^2_{cusp}(GL_n(\mathbb{A}))$ of cusp forms is at most 1.

Question I (Existence)

When does $\otimes' \pi_p$ embeds into $L^2_{cusp.}$? YES

Converse Theorem (Weil, Jacquet, Piatetski-Shapiro, Shalika, Cogdell)

Let $\pi = \otimes' \pi_p$ be an irreducible admissible representation of $GL_n(\mathbb{A})$ $(n \geq 2)$, such that $\mathcal{L}(\pi \times \tau, s)$ is "nice" for every cuspidal representation τ of $GL_m(\mathbb{A})$ $(1 \leq m \leq n-1)$. Then π is a cuspidal representation.

Question II' (strong version)

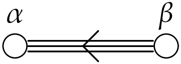
Is a "global" object M uniquely defined by its "local" constituents M_p at all but finitely many primes?

Strong Multiplicity One Theorem (Piatetski-Shapiro, Jacquet, Shalika)

For any two cuspidal representations π and π' of $GL_n(\mathbb{A})$, if π and π' are **nearly equivalent** (i.e. $\pi_p \cong \pi'_p$ for almost all p) then $\pi = \pi'$ **as cuspidal representations.**

The Group G_2

- Let $\mathbb{O}$ denote the (split) Cayley algebra of dimension 8 over $\mathbb{Q}$ named the **octonions**. This is a **non-associative** composition algebra.
- The group of automorphisms of $\mathbb{O}$ is a split, simple, adjoint and simply-connected exceptional group of type G_2.
- There is a natural norm map $\mathrm{Nm} : \mathbb{O} \to \mathbb{Q}$ so G_2 admits a natural embedding to SO_8.
- G_2 preserves the decomposition $\mathbb{O} = \mathbb{Q} \oplus \mathbb{O}^0$, where $\mathbb{O}^0$ are the imaginary octonions and hence G_2 admits an embedding to SO_7 (this is the standard representation of G_2).



Theorem (Gan,Gurevich, Jiang)

- For any finite set of primes S there exists a representation π_S, unramified at all primes not in S, of $G_2(\mathbb{A})$ such that

$$mult_{L^2_{cusp}}(\pi_S) = \frac{1}{6}\left(2^{\#S} + (-1)^{\#S}2\right) - 1.$$

- $\mathcal{L}^S(s, \pi_S)$ has a double pole of order 2 at $s = 2$.

Theorem (S)

Let π be a cuspidal representation of $G_2(\mathbb{A})$ such that $\mathcal{L}^S(s, \pi_S)$ has a double pole of order 2 at $s = 2$. Then $\pi \cong \pi_S$ as abstract representation.

Something About the Proof

Theorem (S) - An Integral Representation

$$\int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \varphi(g)\,\mathcal{E}(s,g)\,dg = \mathcal{L}^S(s,\pi)\,d_S(\varphi,\mathcal{E},s),$$

where $\varphi \in \pi$ and $\mathcal{E}$ is a certain degenerate Eisenstein series on $Spin_8$.

Remark

It can happen that the LHS is 0 (and then so is $d_S(\varphi,\mathcal{E},s)$). However, if $\mathcal{L}^S(s,\pi_S)$ has a double pole of order 2 at $s=2$ then both sides are non-zero.

Corollary

$\mathcal{L}^S(s,\pi)$ admits meromorphic continuation.

- $G_2 \times S_3 \hookrightarrow \mathrm{Spin}_8 \rtimes S_3$ is a dual reductive pair.

Theorem (Gan, Gurevich, Jiang)

For the split $Spin_8$, the Eisenstein series $\mathcal{E}(s, g)$ admits a double pole at $s = 2$ and the residual representation Π is minimal.

- Let $\theta(g) = \mathrm{Res}_{s=2}[\mathcal{E}(s, g)]$.

Remark

$$\Theta_{S_3}(\varphi)(h) = \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \varphi(g)\,\theta(g, h)\,dg, \qquad (g, h) \in G_2(\mathbb{A}) \times S_3(\mathbb{A})$$

generates a representation $\Theta_{S_3}(\pi)$ of $S_3(\mathbb{A})$.

Goal

Show that $\Theta_{S_3}(\pi) \neq 0$.

$$\int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \varphi(g)\,\mathcal{E}(s,(g))\,dg = \mathcal{L}^S(\pi,s)\,d_S(\varphi,\mathcal{E},s),$$

$$\int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \varphi(g)\,\theta(g,h)\,dg = \Theta_{S_3}(\varphi)(h), \qquad (g,h) \in G_2(\mathbb{A}) \times S_3(\mathbb{A}).$$

- Assume that $\mathcal{L}^S(\pi,s)$ admits a double pole at $s=2$, then

$$\begin{aligned} 0 &\neq \lim_{s\to 2}(s-2)^2\,\mathcal{L}^S(\pi,s) \\ \Rightarrow \exists\varphi,\mathcal{E}: \quad 0 &\neq \mathrm{Res}_{s=2}\left[\int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \varphi(g)\,\mathcal{E}(s,g)\,dg\right] \\ &= \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \varphi(g)\,\mathrm{Res}_{s=2}\left[\mathcal{E}(s,g)\right]dg \\ &= \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \varphi(g)\,\theta(g)\,dg = \Theta_{S_3}(\varphi)(1) \end{aligned}$$

- Hence $\Theta_{S_3}(\pi) \neq 0$.

Shadows of Eisenstein Series

Conjecture (Ramanujan-Petersson-Satake)

For any cuspidal representation $\pi = \otimes' \pi_p$ of $G(\mathbb{A})$, π_p is tempered for all p (i.e. $\pi_p \in L^{2+\epsilon}\left(G\left(\mathbb{Q}_p\right)\right)$ for all $\epsilon > 0$).

- A representation π is called **CAP** (cuspidal associated to parabolic) with respect to the Borel subgroup B if there is a character μ of $B(\mathbb{A})$ such that π is nearly-equivalent to an irreducible constituent of $\mathrm{Ind}_{B(\mathbb{A})}^{G_2(\mathbb{A})} \mu$.
- CAP representations provide counter examples to the Ramanujan-Petersson-Satake conjecture.

Theorem (S)

All CAP representation π of $G_2(\mathbb{A})$ with respect to the Borel subgroup are constructed by a lift from a quasi-split form of S_3 and can be characterized by the analytic behaviour of $\mathcal{L}(s, \pi)$ at $s = 2$.

Thank You!

The Gan-Gurevich-Jiang Construction

- $G_2 \times S_3 \hookrightarrow \mathrm{Spin}_8 \rtimes S_3$ is a dual reductive pair.
- In fact,

$$\left\{\begin{matrix}\text{Iso. classes of}\\ \text{Quasi-split}\\ \text{forms of } D_4/\mathbb{Q}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Iso. classes}\\ \text{of étale cubic}\\ \text{algebras over } F\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Iso. classes of}\\ \text{Quasi-split}\\ \text{forms of } S_4/\mathbb{Q}\end{matrix}\right\}$$

$$Spin_8^E \longleftrightarrow E \longleftrightarrow S_E$$

and $G_2 \times S_E \hookrightarrow \mathrm{Spin}_8^E \rtimes S_E$ is a dual reductive pair.

Theorem (S)

If π is a CAP representation of $G_2(\mathbb{A})$ with respect to the Borel subgroup can then $\theta_{S_E}(\pi) \neq 0$ for some E (determined by the Fourier coefficients of π).

- For simplicity we will consider the case relevant to S_3. Namely, we assume that π supports the "split" Fourier coefficient.

- Assume that π is a CAP representation with respect to the Borel subgroup B. Namely, π is nearly-equivalent to a subquotient of $\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\mu$. WALOG, one can assume that $\mu=\mu_1\boxtimes\mu_2$ lies in the positive Weyl chamber, i.e. $\mathfrak{Re}(\mu_2)\le\mathfrak{Re}(\mu_1)\le 2\cdot\mathfrak{Re}(\mu_2)$.
- Fact: If $\mu_1=\mu_2=|\cdot|$ then $\Theta_{S_3}(\pi)\neq 0$.

$$\mathcal{L}^S(s,\pi,\chi)=\mathcal{L}_F^S(s,\mu_1\chi)\,\mathcal{L}_F^S\left(s,\mu_1^{-1}\chi\right)\mathcal{L}_F^S(s,\mu_2\chi)\,\mathcal{L}_F^S\left(s,\mu_2^{-1}\chi\right)\mathcal{L}_F^S\left(s,\frac{\mu_1}{\mu_2}\chi\right)\mathcal{L}_F^S\left(s,\frac{\mu_2}{\mu_1}\chi\right)\mathcal{L}_F^S(s,\chi)$$

Theorem (S)

- For $\mathfrak{Re}(s)>0$: If $\chi^2=\mathbf{1}$ then $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ is holomorphic at $s\neq 1,2$ and

$$ord_{s=1}\left(\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})\right)\le 1,\quad ord_{s=2}\left(\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})\right)\le 2.$$

- If $\chi^2\neq\mathbf{1}$ then $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ is holomorphic for any $\mathfrak{Re}(s)>0$.

- For $\chi=\mu_1\cdot|\cdot|^{-1}$, $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ admits a pole at $s=2$ so $\mu_1^2=\mathbf{1}$ or $\mu_1=|\cdot|$. Similarly for μ_2.

- If $\mu_1 = \mathbf{1}$ then $\mathcal{L}^S(s, \pi, \chi, \mathfrak{st}) = \mathcal{L}^S(s, \chi)^3 \mathcal{L}^S(s, \mu_2\chi)^4$ which admits a pole of order at least 4 at $s = 1$ which brings us to a contradiction. Similarly for μ_2.
- Assume $ord(\mu_1) = ord(\mu_2) = 2$.

$$\mathcal{L}^S(s, \pi, \chi, \mathfrak{st}) = \mathcal{L}_F^S(s, \eta_1\chi)^2 \mathcal{L}_F^S(s, \eta_2\chi)^2 \mathcal{L}_F^S(s, \eta_1\eta_2\chi)^2 \mathcal{L}_F^S(s, \chi).$$

It follows that $\mathcal{L}^S(s, \pi, \mu_1, \mathfrak{st})$ admits a pole of order at least 2 at $s = 1$ which brings us to a contradiction.

- It follows that $\mu_1 = \mu_2 = |\cdot|$ and hence $\Theta_{S_3}(\pi) \neq 0$.