Poles of the Standard $\mathcal{L}$-function and Functorial Lifts for G_2

Avner Segal

Department of Mathematics
University of British Columbia
avners@math.ubc.ca

Square-Integrable Automorphic Representations

- Throughout this talk, G is a split, simple and connected algebraic group defined over $\mathbb{Z}$ (e.g. SL_n, PGL_n, Sp_{2n}, SO_m, G_2, etc.).
- Consider the right-regular (complex) representation of $G(\mathbb{A})$:

$$L^2\left(G(\mathbb{Q})\backslash G(\mathbb{A})\right) = L^2_{disc.}(G) \oplus L^2_{cont.}(G)$$

Example - Groups with Discrete and Continuous Spectrum

- Discrete spectrum: $L^2\left(S^1\right) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\cdot e^{2\pi ixn}$.
- Continuous spectrum: $L^2(\mathbb{Z}) = \int_{S^1}^{\oplus}\left(\mathbb{C}\cdot e^{2\pi ixn}\right)dx$.

- A square-integrable automorphic representation is a unitary subrepresentation $\pi \subset L^2_{disc.}(G)$. Automorphic representations are **embedded** representations!

- We say that a form $\varphi \in L^2\left(G\left(\mathbb{Q}\right)\backslash G\left(\mathbb{A}\right)\right)$ is **cuspidal** if

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi\left(ug\right) du = 0$$

  for any unipotent radical U of any proper parabolic subgroup of G and almost all $g \in G\left(\mathbb{A}\right)$.
- The space of cuspidal forms is denoted by $L^2_{cusp.} \subset L^2_{disc.}\left(G\right)$.

$$L^2\left(G\left(\mathbb{Q}\right)\backslash G\left(\mathbb{A}\right)\right) = L^2_{disc.} \oplus L^2_{cont.} = {\color{red}L^2_{cusp.}} \oplus \underbrace{L^2_{res.} \oplus L^2_{cont.}}_{\text{Can be constructed using Eisenstein series}}$$

- A **cuspidal representation** is a unitary subrepresentation of $L^2_{cusp.}$. Cuspidal representations are **embedded** representations!

Irreducible Representations of $G(\mathbb{A})$

Theorem (Flath)

For an irreducible admissible representation π of $G(\mathbb{A})$ we have $\pi = \otimes'_{p\leq\infty}\pi_p$, where π_p is unramified for almost all p.

- $\pi_p : G\left(\mathbb{Q}_p\right) \to End\left(V_{\pi_p}\right)$ is said to be **unramified** if it contains a non-zero $G\left(\mathbb{Z}_p\right)$-fixed vector.

$$\otimes'_{p\leq\infty}\pi_p = Span_{\mathbb{C}}\left\{\otimes_{p\leq\infty}v_p \;\middle|\; v_p = v_p^0 \text{ for almost all } p\right\}.$$

- $\pi = \otimes'_{p\leq\infty}\pi_p$ and $\pi' = \otimes'_{p\leq\infty}\pi'_p$ are said to be **nearly-equivalent** if $\pi_p \cong \pi'_p$ for almost all p.
- If $\pi_p \cong \pi'_p$ for all p, then $\pi \cong \pi'$. However, for automorphic representations, it is possible that $\pi \ncong \pi'$.

Dual Groups

- For a split, simple and connected algebraic group G, Langlands defined a complex Lie group $G^\vee$ called the **complex dual group of** G**.**
- Examples:

$$G = SL_n \rightsquigarrow G^\vee = PGL_n(\mathbb{C}), \qquad G = PGL_n \rightsquigarrow G^\vee = SL_n(\mathbb{C}), \qquad (A_n)$$

$$G = Sp_{2n} \rightsquigarrow G^\vee = SO_{2n+1}(\mathbb{C}), \quad G = SO_{2n+1} \rightsquigarrow G^\vee = Sp_{2n}(\mathbb{C}), \quad (BC_n)$$

$$G = SO_{2n} \rightsquigarrow G^\vee = SO_{2n}(\mathbb{C}), \qquad (D_n)$$

$$G = G_2 \rightsquigarrow G^\vee = G_2(\mathbb{C}). \qquad (G_2)$$

Satake Parameters

Theorem (Satake, Langlands)

There is a bijection

$$\left\{\begin{matrix}\text{Equivalence classes}\\ \text{of unramified}\\ \text{representations of } G\left(\mathbb{Q}_p\right)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Conjugacy classes}\\ \text{of semi-simple}\\ \text{elements in } G^\vee\end{matrix}\right\}.$$

$$\pi_p \longleftrightarrow t_{\pi_p}$$

Example - $G = SL_2$

Let $G = SL_2$, and let $\chi : B \to \mathbb{C}^\times$ be given by $\chi\begin{pmatrix} t & x \\ & t^{-1}\end{pmatrix} = |t|^s$. The parabolic induction $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi$ admits a unique unramified constituent $\pi_{p,s}$. The Satake parameter of $\pi_{p,s}$ is the conjugacy class of $t_{\pi_{p,s}} = \begin{pmatrix} p^{-s} & \\ & 1\end{pmatrix}$ in $PGL_2(\mathbb{C})$.

Functoriality Principle

Given G, H, $r : H^\vee \to G^\vee$, one expects a map

$$\begin{array}{ccc} L^2_{disc.}(H) & \xrightarrow{r_*} & L^2_{disc.}(G) \\ \tau = \otimes' \tau_p & \longrightarrow & \pi = \otimes' \pi_p \end{array}$$

such that $t_{\pi_p} = r\left(t_{\tau_p}\right)$ for almost every p. Such π is called a **weak functorial lift of τ.**

This gives rise to a map:

$$r_* : \left\{\begin{array}{c} \text{nearly equivalence classes} \\ \text{of irreducible automorphic} \\ \text{representations of } H(\mathbb{A}) \end{array}\right\} \to \left\{\begin{array}{c} \text{nearly equivalence classes} \\ \text{of irreducible automorphic} \\ \text{representations of } G(\mathbb{A}) \end{array}\right\}$$

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Question

Given an irreducible automorphic representation $\pi = \otimes \pi_p$ of $G(\mathbb{A})$, is π in the image of r_*?

Possible Approach

Use an explicit construction

$$\mathcal{B} : L^2_{disc.}(G) \to L^2_{disc.}(H)$$

such that if $\mathcal{B}(\pi) \neq 0$ then for any irreducible representation $\tau \subset \mathcal{B}(\pi)$ it holds that $\pi = r_*(\tau)$. Such τ is called a **backward lift** of π.

Example - θ-lift

- Dual reductive pair: Assume that $G \times H \subset M$ so that $Cent_M(G) = H$ and $Cent_M(H) = G$.
- Example: $G = \widetilde{Sp_{2n}}$ and $H = SO_m$ (or vice versa).
- Assume that $M(\mathbb{A})$ admits a minimal representation $\Pi \subset L^2_{res.}(M)$.
- For $\pi \subset L^2_{cusp.}(G)$ irreducible, consider the representation $\Theta(\pi)$ of $H(\mathbb{A})$ spanned by:

$$\Theta(\varphi)(h) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi(g)\,\theta(g,h)\,dg, \quad \varphi \in \pi,\ \theta \in \Pi.$$

Question

Is $\Theta(\pi) \neq 0$? If so, is π a weak functorial lift?

- For $\rho : G^\vee \to GL_N(\mathbb{C})$ and a finite set of primes $S \ni \infty$ such that π is unramified outside of S, Langlands constructed the partial $\mathcal{L}$-function:

$$\mathcal{L}^S(\pi, s, \rho) = \prod_{p \notin S} det\left(1 - \rho\left(t_{\pi_p}\right) p^{-s}\right)^{-1}.$$

- For $G = GL_1$, $\pi = \mathbf{1}$, $S = \{\infty\}$ and $\rho = \mathbf{1}$: $\mathcal{L}^S(\pi, s, \rho) = \zeta(s)$
- This product converges for $\mathfrak{Re}(s) \gg 0$.

Conjecture (Langlands)

$\mathcal{L}^S(\pi, s, \rho)$ admits a meromorphic continuation to $\mathbb{C}$.

$$H^\vee \xrightarrow{r_*} G^\vee \xrightarrow{\rho} GL_N(\mathbb{C})$$

Baby Example - $\mathcal{L}$-functions and Weak Lifts

If ρ is chosen so that $\rho \circ r = \rho' \oplus \mathbf{1}$ then

$$\mathcal{L}_G^S(r_*(\tau), s, \rho) = \mathcal{L}_H^S(\tau, s, \rho \circ r) = \zeta^S(s)\,\mathcal{L}_H^S(\tau, s, \rho'). \qquad (*)$$

If $\mathcal{L}^S(\tau, s, \rho')$ is holomorphic and non-vanishing at $s = 1$ then $\mathcal{L}^S(r_*(\tau), s, \rho)$ has a simple pole at $s = 1$. Usually, one sees multiple (shifted) ζ-functions and Dirichlet $\mathcal{L}$-functions.

Goal - Proving the Converse (Prototype)

If $\mathcal{L}^S(\pi, s, \rho)$ has a pole at $s = s_0$ of order n then $\pi = r_*(\tau)$ for some automorphic representation τ of $H(\mathbb{A})$.

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Main Tool - Integral Representations of $\mathcal{L}$-functions

$$\mathcal{Z}(s, \varphi, \ldots) = \mathcal{L}^S(\pi, s, \rho) \cdot \boxed{\text{ramified factor}},$$

Example

$$\mathcal{Z}(s, \varphi, \mathcal{E}) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi(g)\, \mathcal{E}(g, s)\, dg,$$

where $\varphi \in \pi$ and $\mathcal{E}$ is an Eisenstein series on a larger group G'.

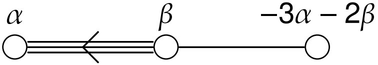
General Strategy (Rallis)

- Assume that $\mathcal{L}^S(\pi, s, \rho)$ admits a pole of order n at $s = s_0$.
- Then $(s - s_0)^n \mathcal{Z}(\varphi, s, ...) \not\equiv 0$ on π.
- Show that $\mathcal{B}(\pi) \neq 0$ (these requires that the construction of $\mathcal{Z}(\varphi, s, ...)$ will encode $\mathcal{B}(\pi)$).
- Let $\tau \subseteq \mathcal{B}(\pi)$ be an irreducible automorphic representation. Show that $\pi = r_*(\tau)$.

- Kudla-Rallis implemented this strategy for $\left(\widetilde{Sp_{2n}}, SO_m\right)$.
- Today: Implementation for $G = G_2$ and various H and r.

The Group G_2

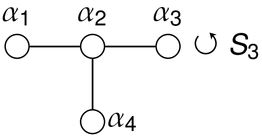
- Let $\mathbb{O}$ denote a (split) Cayley algebra of dimension 8 over $\mathbb{Q}$ named the **octonions**. This is a **non-associative** composition algebra.
- The group of automorphisms of $\mathbb{O}$ is a split, simple, adjoint and simply-connected exceptional group of type G_2.
- There is a natural map $\mathrm{Nm} : \mathbb{O} \to \mathbb{Q}$ so G_2 admits a natural embedding to SO_8.
- In fact, G_2 preserves the decomposition $\mathbb{O} = \mathbb{Q} \oplus \mathbb{O}^0$, where $\mathbb{O}^0$ are the imaginary octonions and hence G_2 admits an embedding $\mathfrak{st}$ into SO_7.



$$r_1 : SL_3(\mathbb{C}) \to G_2(\mathbb{C})$$
$$r_2 : SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to G_2(\mathbb{C})$$

$$\left\{\begin{array}{c}\text{Iso. classes}\\ \text{of étale cubic}\\ \text{algebras over } \mathbb{Q}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Iso. classes of}\\ \text{Quasi-split}\\ \text{forms of } D_4/\mathbb{Q}\end{array}\right\}$$

$$E \longleftrightarrow H_E = Spin_8^E$$



- $(Spin_8)^{S_3} = G_2$.
- In fact, $H_E^{S_E} = G_2$, where $S_E = \mathrm{Aut}_{\mathbb{Q}}(E)$.
- $G_2 \times S_E \subset H_E \rtimes S_E$ is a dual reductive pair.

A Rankin-Selberg Integral

- Let $\pi \boxtimes \chi$ be a cuspidal representation of $G_2(\mathbb{A}) \times GL_1(\mathbb{A})$ and let $\mathcal{E}(\chi, f, s, g)$ be a (normalized) degenerate Eisenstein series on $Spin_8^E(\mathbb{A})$ (induced from the Heisenberg parabolic subgroup).

Theorem (S)

For $\varphi \in \pi$:

$$\begin{aligned} \mathcal{Z}(\chi, \varphi, f, s) &= \int\limits_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \varphi(g)\, \mathcal{E}\left(\chi, f, s - \frac{1}{2}, g\right) dg \\ &= \mathcal{L}^S(s, \pi \boxtimes \chi, \mathfrak{st})\, d_S(f_S, \varphi_S, s). \end{aligned}$$

For any s_0 there exist data so that $d_S(f_S, \varphi_S, s)$ is holomorphic and non-zero.

Both $\mathcal{L}^S(s, \pi \boxtimes \chi, \mathfrak{st})$ and $d_S(f_S, \varphi_S, s)$ admits meromorphic continuation to $\mathbb{C}$. $d_S(f_S, \varphi_S, s)$ is non-zero when E is matched with a non-zero Fourier-coefficient of π.

$$\left\{\begin{array}{c}\text{Fourier coefficients}\\ \text{along the}\\ \text{Heisenberg}\\ \text{subgroup of } G_2\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Iso. classes}\\ \text{of étale cubic}\\ \text{algebras over } F\end{array}\right\} \underset{}{\overset{\text{only for } E \text{ Galois}}{\longleftrightarrow}} \left\{\begin{array}{c}\text{Hecke characters}\\ \chi : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times\\ \text{of order } 1,2,3\\ \text{up to conjugation}\end{array}\right\}$$

$$E \longleftrightarrow \chi_E$$

Corollary

$$\mathrm{ord}_{s=s_0} \mathcal{L}^S(s,\pi,\chi,\mathfrak{st}) \leq \mathrm{ord}_{s=s_0} \mathcal{E}\left(\chi, s-\tfrac{1}{2}, f, g\right).$$

The Poles of $\mathcal{E}_E\left(\chi, s-\frac{1}{2}, f, g\right)$ for $\mathfrak{Re}(s) > 0$ (S)

	$s=1$	$s=2$		$s=3$
	$\chi^2=1$	$\chi=1$	$\chi=\chi_E$	$\chi=1$
$E=F\times F\times F$	1	2		1
$E=F\times K$	1	1	1	1
E Galois field extension	1	0	1	1
E non-Galois	1	0	╳	1

Backward Lift for r_{1*}: The Gan-Gurevich-Jiang Lift

- (Gan, Gurevich, Jiang) The residual representation of $\mathcal{E}\left(\chi_E, f, s-\frac{1}{2}, g\right)$ at $s=2$ is the minimal representation Π_E.
- The dual reductive pair $G_2 \times S_E$ in $Spin_8^E \rtimes S_E$ gives rise to an exceptional Θ-lift.
- Assume that $\mathcal{L}^S\left(s, \pi, \chi_E, \mathfrak{st}\right)$ has a pole of maximal order at $s=2$.

$$\begin{aligned} 0 \neq \operatorname{Res}_{s=2}\left[\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi(g)\,\mathcal{E}(\chi, f, s, g)\,dg\right] \\ = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi(g)\operatorname{Res}_{s=2}\left[\mathcal{E}(\chi, f, s, g)\right] dg \\ = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi(g)\,\theta(g)\,dg = \Theta_{S_E}(\varphi)(1) \end{aligned}$$

Theorem (S)

IF $\mathcal{L}^S\left(s, \pi, \chi_E, \mathfrak{st}\right)$ admits a pole of maximal order at $s=2$ then $\Theta_{S_E}(\pi) \neq 0$.

So what did that had to do with r_{1*}?

General Theory	GGJ-lift
G	$GL_1 \times G_2$
$G^\vee$	$\mathbb{C}^\times \times G_2(\mathbb{C})$
H	PGL_3, $PD^\times$
$H^\vee$	$SL_3(\mathbb{C})$
$r : H^\vee \to G^\vee$	$r_1 : SL_3(\mathbb{C}) \to G_2(\mathbb{C})$
$\rho : G^\vee \to GL_N(\mathbb{C})$	$\rho = \mathfrak{st} : \mathbb{C}^\times \times G_2(\mathbb{C}) \to GL_7(\mathbb{C})$
"$\mathrm{ord}_{s=s_0} \mathcal{L}^S(s,\pi,\rho) = n$ $\Rightarrow \exists\tau : r_{1*}(\tau) = \pi$"	$\mathrm{ord}_{s=2} \mathcal{L}^S(s,\pi,\chi_E,\mathfrak{st}) = n_E$ $\Rightarrow \Theta_{S_E}(\pi) \neq 0$ $\overset{Gan}{\Rightarrow} r_{1*}(\chi_E) = \pi, \quad$ when $\chi_E^3 = \mathbf{1}$

Remark

All automorphic representations of $S_E(\mathbb{A})$ are nearly equivalent to the trivial representation.

Backward Lift for r_{2*}: The Rallis-Schiffmann Lift

- For a square-integrable irreducible representation σ of $\widetilde{SL_2}(\mathbb{A})$, the theta lift $\theta_{14}(\sigma)$ to SO_7 is irreducible and non-cuspidal.
- If σ is Saito-Kurokawa ($\theta_{10}(\sigma) \neq 0$) then the restriction of $\theta_{14}(\sigma)$ to $G_2(\mathbb{A})$ is a non-zero cuspidal representation (not necessarily irreducible).

Theorem (Gurevich-S)

For a cuspidal irreducible representation π of $G_2(\mathbb{A})$ the following is equivalent

1. There exists an automorphic irreducible square integrable representation σ of $\widetilde{SL_2}(\mathbb{A})$ such that π is a weak lift of σ.
2. The partial $\mathcal{L}$-function $\mathcal{L}^S(s, \pi, \mathbf{1}, st)$ has a pole at $s = 2$.

The pole of $\mathcal{L}^S(s, \pi, \mathbf{1}, st)$ is simple unless $\Theta_{S_3}(\pi) \neq 0$

- Assume that $\mathcal{L}^S(s, \pi, \mathbf{1}, st)$ has a simple pole at $s = 2$.
- It follows that π supports an $F \times K$-Fourier coefficient, where $K \neq F \times F$.
- It follows that there exist $\varphi \in \pi$ and $\mathcal{E}(\mathbf{1}, s, f, g)$

$$\begin{aligned} 0 &\neq \mathrm{Res}_{s=2}\left[\int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \varphi(g)\,\mathcal{E}(\mathbf{1}, s, f, g)\, dg\right] \\ &= \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \varphi(g)\,\mathrm{Res}_{s=2}\left[\mathcal{E}(\mathbf{1}, s, f, g)\right] dg, \end{aligned}$$

Theorem - Siegel-Weil Type Identity

$$\mathrm{Res}_{s=2}\left[\mathcal{E}_{F\times K}(\mathbf{1}, s, f, g)\right] = \widetilde{\mathcal{E}}\left(s, \widetilde{f}, g\right)\Big|_{s=2},$$

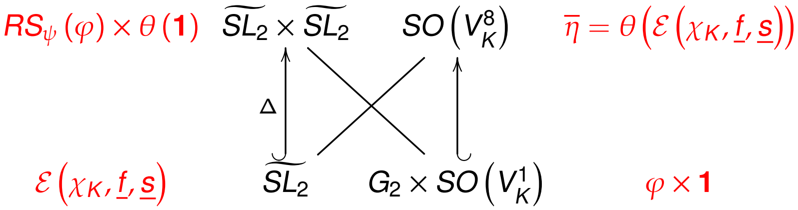
where $\widetilde{\mathcal{E}}$ is a degenerate Eisenstein series associated to the Siegel parabolic subgroup Q of $H_{F\times K}$.

- Let Π_Q denote the residual representation of $\widetilde{\mathcal{E}}$.
- It follows that there exist $\eta \in \Pi_Q$ such that

$$\begin{aligned} 0 &\neq \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \varphi(g)\,\eta(g)\,dg \\ &= \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \varphi(g)\,\overline{\eta}(g)\,dg, \end{aligned}$$

  where $\overline{\eta}$ is in the residual representation $\Pi_{\overline{Q}}$ of a degenerate Eisenstein series in $SO\left(V_K^8\right)$ - the orthogonal group of an 8-dimensional quadratic space with discriminant associated to K.
- Note that $V_K^8 = V_{split}^7 \oplus V_K^1$ and that $G_2 \subset SO\left(V_{split}^7\right)$.

- Consider the following see-saw diagram in $\widetilde{Sp}_{16}$



$$\int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \varphi(g)\,\theta\left(\mathcal{E}\left(\chi_K, \underline{f}, \underline{s}\right)\right)(g)\,dg \neq 0$$
$$\Rightarrow \int_{\widetilde{SL}_2(\mathbb{Q})\backslash \widetilde{SL}_2(\mathbb{A})} RS_\psi(\varphi)(h)\,\theta(\mathbf{1})(h)\,\mathcal{E}\left(\chi_K, \underline{f}, \underline{s}, h\right) dh \neq 0$$

- We conclude that $RS_\psi(\varphi) \neq 0$.

So what did that had to do with r_{2*}?

General Theory	RS-lift
G	G_2
$G^\vee$	$G_2(\mathbb{C})$
H	$PGL_2 \times PGL_2$
$H^\vee$	$SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$
$r : H^\vee \to G^\vee$	$r_2 : SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to G_2(\mathbb{C})$
$\rho : G^\vee \to GL_N(\mathbb{C})$	$\rho = \mathfrak{st} : G_2(\mathbb{C}) \to GL_7(\mathbb{C})$
"$\mathrm{ord}_{s=s_0} \mathcal{L}^S(s,\pi,\rho) = n$ $\Rightarrow \exists\tau : r_{1*}(\tau) = \pi$"	$\mathrm{ord}_{s=2} \mathcal{L}^S(s,\pi,\mathfrak{st}) = 1$ $\Rightarrow RS_\psi(\pi) \neq 0$ $\overset{RS}{\Rightarrow} r_{2*}(\tau \boxtimes \mathbf{1}) = \pi, \quad \tau = Wd\left(RS_\psi(\pi)\right)$

Thank You!