Ben-Gurion University of the Negev
The Faculty of Natural Sciences
Department of Mathematics

Towards a New Integral Representation With No Unique Model of the Standard $\mathcal{L}$-function of a Cuspidal Automorphic Representation of G_2

Thesis Submitted in Partial Fulfilment of the Requirements
for the Master of Sciences Degree

By: Avner Segal

Under the Supervision of: Dr. Nadya Gurevich

Beer Sheva, October 2011

In memory of my father

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Signature of Supervisor: ________________ Date: ________
Signature of Chairperson of the Committee for Graduate Studies: ________________ Date: ________

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Abstract

Let $\mathcal{L}^S(\pi, s, \mathfrak{st})$ be a partial $\mathcal{L}$-function of degree 7 of a cuspidal automorphic representation π of the exceptional group G_2. D. Ginzburg established the meromorphic continuation of $\mathcal{L}^S(\pi, s, \mathfrak{st})$ for generic π by constructing its Rankin-Selberg integral representation. In this thesis we work toward a Rankin-Selberg integral for not necessarily generic representations having certain Fourier coefficient.

The candidate global zeta integral, suggested by Dihua Jiang, is not factorizable since the involved model is not unique. Inspired by the ideas of I. Piatetski-Shapiro and S. Rallis we make progress toward showing that the local integral applied to a spherical vector in any model produces the standard L-factor. We hope to apply our method to other Rankin-Selberg integrals which unfold to non-unique models.

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Contents

Introduction

0.1 History and Open Questions

$\mathcal{L}$-functions are useful tools for obtaining algebraic information about a group and its representations using analytic enquiries. The literature contains a few remarkable examples of such uses. The first usage of $\mathcal{L}$-functions dates back to 1837 when Dirichlet defined and used them to prove that in any arithmetic progression $an+b$, with a and b coprime, there are infinitely many primes.

The most basic reason to be interested in $\mathcal{L}$-functions is portrayed in [Bom05]. One can attach $\mathcal{L}$-functions to automorphic representations, Galois representations, algebraic varieties and other arithmetical objects. The philosophy is that two objects with equal $\mathcal{L}$-functions, each from a different category, are somehow connected.

The connection between p-adic or automorphic representations and Galois representations is the heart of the Langlands program. It is conjectured that corresponding representations will have the same $\mathcal{L}$-functions. The Langlands conjectures are portrayed in [Kna97].

In order to obtain information about $\mathcal{L}$-functions one needs to establish their meromorphic continuation. This problem is far from being solved. One way to address this problem, called the *Rankin-Selberg method*, is by attaching an integral representation to the $\mathcal{L}$-function. The method seeks to construct automorphic $\mathcal{L}$-functions as integrals of automorphic forms. The Rankin-Selberg representation also gives information about special values of $\mathcal{L}$-functions.

The first example of a Rankin-Selberg integral was introduced in 1939 by R. Rankin [Ran39] when he constructed an integral representation for the tensor product $\mathcal{L}$-function of $GL(2) \times GL(2)$.

As already mentioned, the problem of giving an integral representation to $\mathcal{L}$-functions has not been solved in general but has been solved for many cases. In most of the known cases, the proof that a global integral is an Eulerian product relies on attaching to representations of the group a unique model that factors to a product of local ones.

Jacquet and Langlands in [JL70] used uniqueness of a Whittaker model in the case of $\mathrm{GL}(2)$ and this can be extended to any $\mathrm{GL}(n)$. In [Gin93], D. Ginzburg also used the Whittaker functional when he established the meromorphic continuation of the standard $\mathcal{L}$-function of a generic representation of G_2. Godement and Jacquet used uniqueness of a G-invariant bilinear form for $\mathrm{GL}(n)$. This method was extended by Piatetski-Shapiro and Rallis [GPSR87] to all classical groups in a method known as *doubling of variables*.

Sometimes, the global integrals unfold with nonunique models. In [PSR88] Piatetski-Shapiro and Rallis suggested that such a uniqueness is not needed to show the factorizability of the global integral. Their integral unfolded to a nonunique model, but the local integral applied to a spherical vector in any model of the representation produced the same local factor. Our goal is to follow their ideas in the case of the split exceptional group G_2 in order to find Rankin-Selberg representation for the 7-degree standard $\mathcal{L}$-function.

0.2 Goals and Consequences

For an irreducible cuspidal automorphic representation π of $\mathrm{G}_2(\mathbb{A})$ and a certain Eisenstein series $E(f,s,g)$ of $Spin_8$, we form the following global zeta integral

$$\zeta\left(s,\phi_{\pi},f\right)=\int_{G_2(k)\backslash G_2(\mathbb{A})}E\left(f,s,g\right)\phi_{\pi}\left(g\right)dg\,. \tag{1}$$

In this paper we work toward a proof of

Conjecture 0.1 (Main Conjecture). *Let π be an irreducible cuspidal automorphic representation of $G_2\left(\mathbb{A}\right)$ that has a non-vanishing split Fourier coefficient along the Heisenberg unipotent subgroup. For any factorizable data $\phi_\pi\in V_\pi$ and a section f_s there exists a finite set of places S such that*

$$\zeta\left(s,\phi_{\pi},f\right)=\mathcal{L}^S\left(\pi,5s-2,\mathfrak{st}\right)\frac{d_S\left(s,f_S,\phi_{\pi_S}\right)}{j_S\left(s\right)}\quad\forall s\in\mathbb{C}\,, \tag{2}$$

where $d_S\left(s,f_S,\phi_{\pi_S}\right)$ is a meromorphic function of s and $j_S\left(s\right)$ is the normalizing factor of $E\left(f,s,g\right)$.

Moreover, for any $s_0\in\mathbb{C}$ the data $\phi_{\pi,v}$ and $f_{s,v}$ for $v\in S$ can be chosen so that $d_S\left(s,f_S,\phi_{\pi_S}\right)$ is holomorphic and non-vanishing in a neighbourhood of s_0.

The notion of a split Fourier coefficient is introduced in section 1.3.3. The strategy of the proof of this conjecture is outlined in chapter 4.

The Eisenstein series we work with has at most a double pole at $s=\frac{4}{5}$ and hence the $\mathcal{L}$-function has at most a double pole at $s=2$. The residue of the Eisenstein series at $s=\frac{4}{5}$ generates the minimal representation Π of $Spin_8$. Taking the residue at $\frac{4}{5}$ give the following result.

Corollary 0.1. *The following statements are equivalent*

1. $\mathcal{L}^S\left(\pi,s,\mathfrak{st}\right)$ *has a double pole at* $s=2$
2. $\int_{G_2(k)\backslash G_2(\mathbb{A})}\phi_\pi\left(g\right)F\left(g\right)dg\neq 0$ *for some* $\phi_\pi\in V_\pi,\ F\in\Pi$.

The integral in (2) is non-vanishing if and only if the image of π under the θ-lift for the dual pair (G_2,S_3), is not zero. It was proven in [GGJ02] that such π satisfies

- π appears in the discrete spectrum of G_2.

- π has a non-vanishing split Fourier coefficient along the Heisenberg unipotent subgroup.
- $\mathcal{L}^S(\pi, s, \mathfrak{st})$ has a pole at $s = 2$ of order 2.

Assuming the conjecture, taking the residue of the zeta integral at $s = \frac{4}{5}$ we obtain the converse:

Corollary 0.2. *Let π be an irreducible cuspidal representation of G_2 with a non-vanishing split Fourier coefficient. If $\mathcal{L}^S(\pi, s, \mathfrak{st})$ has a double pole at $s = 2$ then $\theta(\pi) \neq 0$.*

0.3 The Content

This paper is structured as follows:

- Chapter 1 introduces the notion of a Chevalley group and the groups G_2 and $Spin_8$ in particular together with their properties.
- Chapter 2 is divided into two parts. In the first we list some properties of unramified representations of a reductive group over a local field. We also introduce the Hecke algebra and the Satake isomorphism. The second part defines cuspidal automorphic representations and their partial $\mathcal{L}$-functions.
- Chapter 3 gives some basic information about a certain Eisenstein series of $Spin_8$.
- Chapter 4 introduces the global zeta integral. We derive the conjecture from the unfolding and the local computation. In the last part of this chapter we outline the strategy of the local computation. All central ideas of the proof of the main theorem are presented in this chapter.
- Chapter 5 proves the unfolding of the global integral.

- Chapter 6 and 7 are devoted to local computations.

This thesis does not contain the complete proof of the conjecture. We reduce the problem to an equality between two functions on G_2 taking values in the ring of rational functions $\mathbb{C}(q^{-s})$. These two functions are independent of the representation. We verify the equality of the functions at the identity element, which gives a strong evidence of equality of the functions themselves. The proof of this equality was completed after the submission of the thesis, and hence the equality in Conjecture 4.1.

Chapter 1

Chevalley Groups

1.1 Construction of Chevalley Groups

We will present the construction of Chevalley groups and some of their basic properties. However, before beginning that we will introduce some definitions and notations used later in this chapter and work. A more detailed exposition to this matter can be found in [CKM04, Kim's lectures, Chapter I]. Throughout this section we let F be a field of characteristic zero.

1.1.1 Definitions, Notations and Basic Results

A *linear algebraic group* G is an affine group scheme embedded as a Zariski-closed subgroup in GL_n. We say that G is defined over F if this embedding is defined over F. This means that $G(F)$ is the set of F-solutions of a set of polynomial equations with coefficients in F. From now on we simply assume that G is embedded in some GL_n and fix this embedding.

Remark 1.1. Note that $G(F)$ comes with a natural action on F^n.

An element $u \in G$ is called *unipotent* if $u - \mathbb{1} \in M_n$ is nilpotent. A subgroup N of G is called unipotent if any element of N is such. An element $g \in G$ is called *semi-simple* if it is similar to a diagonal element over

$\bar{F}$.

Let G be a connected algebraic group; the *radical* of G, denoted by $R(\mathrm{G})$, is the maximal connected solvable normal subgroup of G. The *unipotent radical* $R_u(\mathrm{G})$ of G is the subgroup of unipotent elements of $R(\mathrm{G})$. The group G is called *semi-simple* if $R(\mathrm{G}) = \{1\}$ and is called *reductive* if $R_u(\mathrm{G}) = \{1\}$.

By Jordan decomposition, any element $g \in \mathrm{G}$ can be written uniquely as $g = g_s g_u$ where g_s is semi-simple and g_u is unipotent and they commute.

Remark 1.2. An important fact about unipotent and reductive groups is that they are unimodular, i.e. a right Haar measure is also a left Haar measure.

A subgroup $\mathrm{T} \subset \mathrm{G}$ is called a *torus of rank n* if it is isomorphic to $\mathbb{G}_m^n$. A torus T is called F-*split* if the isomorphism is defined by a polynomial with coefficients in F. A group G is called F-split if it contains a maximal torus which is F-split. The rank of the group G is defined to be the rank of its maximal torus.

Remark 1.3. From now on we assume that G is a linear algebraic reductive split group defined over F. We also fix a maximal torus T of G.

We denote the group of algebraic *characters* of T to be $X^*(\mathrm{T}) = \mathrm{Hom}_{Alg}(\mathrm{T}, \mathbb{G}_m)$. It is clear that $X^*(\mathbb{G}_m^n) \cong \mathbb{Z}^n$. The group of algebraic *cocharacters* of T is $X_*(\mathrm{T}) = \mathrm{Hom}_{Alg}(\mathbb{G}_m, \mathrm{T})$.

There is a natural pairing $<,>: X^*(\mathrm{T}) \times X_*(\mathrm{T}) \longrightarrow \mathbb{Z}$, for $\chi \in X^*(\mathrm{T})$ and $\mu \in X_*(\mathrm{T})$. The composition $\chi \circ \mu : \mathbb{G}_m \longrightarrow \mathbb{G}_m$ is algebraic and hence of the form $\chi \circ \mu(t) = t^k$. We define $< \chi, \mu >= k$. This pairing induces an isomorphism of $\mathbb{Z}$-modules between $X_*(\mathrm{T})$ and $\mathrm{Hom}(X^*(\mathrm{T}), \mathbb{Z})$.

We will now define the *Lie algebra* $\mathfrak{g}$ of G, $\mathfrak{g}$ is also denoted by Lie (G). Let t be a symbol[1] such that $t^2 \equiv 0$. We define $\mathfrak{g} = \{X \in \mathrm{M}_n(\mathrm{F}) | 1 + tX \in \mathrm{G}[F[t]]\}$, which is a Lie algebra in the usual sense. The dimension of $\mathfrak{g}$ as a vector space equals the dimension of G as a variety.

We have the adjoint representation of G defined by $Ad(g)X = gXg^{-1}$. Let T be a maximal torus in G, in which case $Ad(\mathrm{T})$ is a set of commuting, simultaneously diagonalizable endomorphisms of $\mathfrak{g}$.

[1]one may think of $\mathrm{M}_n(\mathrm{F})[t]/t^2$

Proposition 1.1. *We have a decomposition*

$$\mathfrak{g} = \mathfrak{g}_0^{(\mathrm{T})} \oplus \oplus_{\alpha \in X^*(\mathrm{T})} \mathfrak{g}_\alpha^{(\mathrm{T})}$$

where for $\alpha \in X^*(\mathrm{T})$ *with* $\alpha \neq 0$ *we have*

$$\mathfrak{g}_\alpha^{(\mathrm{T})} = \{X \in \mathfrak{g} | \forall t \in \mathrm{T} : \ Ad(t) X = \alpha(t) X\}$$

Also, there are only finitely many such α *with* $\mathfrak{g}_\alpha^{(\mathrm{T})} \neq 0$*. For* $\alpha = 0$ *we have*

$$\mathfrak{g}_0^{(\mathrm{T})} = \{X \in \mathfrak{g} | \forall t \in \mathrm{T} : \ Ad(t) X = X\} \supset \mathrm{Lie}(\mathrm{T})$$

Also, given $\alpha, \beta \in \Phi$*,* $X \in \mathfrak{g}_\alpha^{(\mathrm{T})}$ *and* $Y \in \mathfrak{g}_\beta^{(\mathrm{T})}$ *then* $[X, Y] \in \mathfrak{g}_{\alpha+\beta}^{(\mathrm{T})}$*.*

A character α such that $\mathfrak{g}_\alpha^{(\mathrm{T})} \neq 0$ is called a *root*. We denote the set of roots of T with respect to G by Φ. We denote $\mathfrak{h} := \mathfrak{g}_0^{(\mathrm{T})}$, which is a Cartan subalgebra.

Given $\alpha, \beta \in \Phi$, $X \in \mathfrak{g}_\alpha^{(\mathrm{T})}$ and $Y \in \mathfrak{g}_\beta^{(\mathrm{T})}$ if $\alpha + \beta \notin \Phi$ then $[X, Y] = 0$.

The centralizer of T in G is T itself. We define the *Weyl group* of T with respect to G to be $W = N_{\mathrm{G}}(\mathrm{T}) / \mathrm{T}$; this is a finite group. The natural action of W on T induces an action on $X^*(\mathrm{T})$ given by:

$$\forall w \in \mathrm{W}, \chi \in X^*(\mathrm{T}), g \in \mathrm{G} : \ {}^{w}\chi(g) = \chi\left(w^{-1} g w\right)$$

We construct a set $\Phi^\vee \subset X_*(\mathrm{T})$, dual to Φ, called the set of *coroots* of T with respect to G. By construction it is in bijection with Φ.

Given $\alpha \in \Phi$ the connected component of the identity of $\ker(\alpha)$ satisfies $(\ker(\alpha))^0 \subset \mathrm{T}$. $Z_{\mathrm{G}}\left((\ker(\alpha))^0\right)$ is a connected reductive group with T as a maximal torus. Let

$$\mathrm{G}_\alpha = \left[Z_{\mathrm{G}}\left((\ker(\alpha))^0\right), Z_{\mathrm{G}}\left((\ker(\alpha))^0\right)\right].$$

This is isomorphic to either $\mathrm{SL}(2)$ or $PGL(2)$. G_α has a maximal torus $T_\alpha \subset \mathrm{T}$. We define $\alpha^\vee : \mathbb{G}_m \longrightarrow T_\alpha$ to be the unique homomorphism such that $< \alpha, \alpha^\vee >= 2$.

The quadrupole $(X^*(\mathrm{T}), \Phi, X_*(\mathrm{T}), \Phi^\vee)$ is called a *root datum* of G. Also, the triple $\left(X^*\left(\mathrm{T} / Z(\mathrm{G})^0\right), \Phi, \mathrm{W}\right)$ is called a *root system*.

There is a natural embedding of Φ in W. For any $\alpha \in \Phi$ we define w_α which acts on $X^*(\mathrm{T})$ by

$$w_\alpha(\chi) = \chi - <\chi, \alpha^\vee> \alpha \; .$$

We note that $w_\alpha(\alpha) = -\alpha$.

We are going to realize the roots and coroots in a real vector space. We consider $X_{\mathbb{R}} := X^*(\mathrm{T}) \otimes_{\mathbb{Z}} \mathbb{R}$. Take any inner product $[,]$ on $X_{\mathbb{R}}$. We define $(x, y) = \frac{1}{|W|} \sum_{w \in W} [wx, wy]$ which is a W-invariant positive definite symmetric bilinear form on $X_{\mathbb{R}}$.

From $(w_\alpha(\chi), w_\alpha(\chi)) = (\chi, \chi)$ and $w_\alpha(\chi) = \chi - <\chi, \alpha^\vee> \alpha$ we deduce that $<\chi, \alpha^\vee> = \frac{2(\alpha,\chi)}{(\alpha,\alpha)}$. We identify $\alpha^\vee$ with $\frac{2\alpha}{(\alpha,\alpha)}$.

The element w_α fixes a hyperplane $H_\alpha = \{\chi \in X_{\mathbb{R}} | (\alpha, \chi) = 0\}$ in $X_{\mathbb{R}}$. $X_{\mathbb{R}} \setminus \cup_{\alpha \in \Phi} H_\alpha$ is a finite union of connected components called *Weyl chambers*. The Weyl group acts simply transitively on the set of Weyl chambers[2]. We choose a Weyl chamber C, which gives rise to a linear order on X.

We say that $\alpha > 0$ if $(\alpha, \chi) > 0$ for all $\chi \in C$. We denote the positive roots by Φ_+. A positive root α is called *simple* if it cannot be written as $\beta + \gamma$ with $\beta, \gamma \in \Phi_+$. We denote the set of simple roots by Δ. We denote the elements of Δ by $\alpha_1, \ldots, \alpha_l$.

Theorem 1.1. *The following holds*

1. *$l = rank\,(\mathrm{T})$.*

2. *Every root $\alpha \in \Phi$ is of the form*

$$\alpha = \pm \sum_{i=1}^{l} m_i \alpha_i \; ,$$

with $m_i \in \mathbb{N} \cup \{0\}$

3. *W is generated by the w_{α_i}*

[2]All chambers are in the same orbit but the stabilizer of each chamber is trivial

4. *Every root $\alpha \in \Phi$ can be written in the form*

$$\alpha = w_{\alpha_{i_r}} \cdots w_{\alpha_{i_1}} \alpha_{i_0} ,$$

with $\alpha_{i_r}, \ldots, \alpha_{i_0} \in \Delta$.

5. *If α, β are non-proportional roots and $(\alpha, \beta) > 0$ then $\alpha - \beta \in \Phi$.*

For $\alpha, \beta \in \Phi$ we define $c_{\alpha,\beta} = \langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$, we call this number a *Cartan integer*. For two simple roots α_i, α_j the Cartan integer is 2 if $i = j$ and can be either $0, -1, -2$ or -3 if $i \neq j$. The *Cartan matrix* of Φ is defined to be

$$C = \left(c_{\alpha,\beta}\right)_{\alpha,\beta\in\Phi} .$$

A root system Φ is called irreducible if it cannot be written as $\Phi = \Phi_1 \coprod \Phi_2$ where Φ_1, Φ_2 are non-empty subsystems such that $\Phi_1 \perp \Phi_2$ (in the sense of the W-invariant bilinear form $(,)$).

Theorem 1.2. *There exists a one to one correspondence between irreducible root systems and simple Lie algebras*

We denote

$$X^+ = \{\chi \in X_{\mathbb{Q}} \big| (\chi, \alpha^\vee) \in \mathbb{Z}\ \forall \alpha^\vee \in \Phi^\vee\} .$$

This lattice is called the *weight lattice*. The group G is called *simply connected* if $X^*(T) = Span_{\mathbb{Z}}(\Delta)$. The group G is called *adjoint* if $X^*(T) = X^+$. In this case G has a trivial center.

One can define an abstract notion of root systems and classify all irreducible root systems in terms of Dynkin diagrams. We omit this discussion since we are interested only in two such types of root systems which will be discussed later.

1.1.2 Construction of Chevalley Groups and Their Structure

Let Φ be a root system and let $\mathfrak{g}$ be a semi-simple Lie algebra determined by Φ. Hence

$$\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha\in\Phi} \mathfrak{g}_\alpha ,$$

where $\dim(\mathfrak{g}_\alpha) = 1$ for any $\alpha \in \Phi$. Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ and $l = \dim \mathfrak{h}$. For each $\alpha \in \Phi$ let $H'_\alpha \in \mathfrak{h}$ be such that $(H, H'_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}$; the existence of such an H'_α is due to Riesz theorem. We define $H_\alpha = \frac{2}{(\alpha,\alpha)} H'_\alpha$ and we write H_i for H'_{α_i}.

Theorem 1.3 (Existence of Chevalley basis)**.** *Given $H_i, i = 1, \ldots, l$ chosen as above, one can find for each $\alpha \in \Phi$ an element $0 \neq E_\alpha \in \mathfrak{g}_\alpha$ such that H_i, E_α form a basis of $\mathfrak{g}$ satisfying the following equations of structure:*

1. $\left[H_i, H_j\right] = 0$, *i.e.* $\mathfrak{h}$ *is commutative.*
2. $[H_i, E_\alpha] = c_{\alpha_i,\alpha} E_\alpha$.
3. $\left[E_\alpha, E_\beta\right] = \begin{cases} 0, & \beta = \alpha \\ H_\alpha, & \beta = -\alpha \\ n_{\alpha,\beta} E_{\alpha+\beta}, & \alpha + \beta \in \Phi \\ 0, & \alpha + \beta \notin \Phi \end{cases}.$

Remark 1.4. Most sources tend to be more specific about the value of the structure constants $n_{\alpha,\beta}$, but they depend on the choice of E_α and we will use different values than customary.

The elements E_α act nilpotently on $\mathfrak{g}$ by the adjoint action and thus for any $\alpha \in \Phi$ and any $t \in \mathrm{F}$ the exponential map is well defined for E_α. We define

$$x_\alpha(t) = exp(tE_\alpha) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n E_\alpha^n \in GL(\mathfrak{g}) .$$

We denote $U_\alpha = \{x_\alpha(t) | t \in \mathrm{F}\}$. The group generated by U_α for all α is a simply connected group called the *Chevalley group*.

For $t \in \mathrm{F}$ we let

$$w_\alpha(t) = x_\alpha(t)\, x_{-\alpha}\left(-t^{-1}\right) x_\alpha(t)$$
$$h_\alpha(t) = w_\alpha(t)\, w_\alpha(-1) .$$

Let U be the subgroup of G generated by U_α for all $\alpha \in \Phi_+$. Let T be the subgroup of G generated by all $h_\alpha(t)$ with $\alpha \in \Phi$ and $t \in \mathrm{F}$. This is a maximal torus.

We can reduce many calculations in G to calculations in SL (2) using the following proposition.

Proposition 1.2. *If* $\alpha \in \Phi$, *there exists an injective homomorphism* $\phi_\alpha : \mathrm{SL}(2) \longrightarrow \mathrm{G}$ *such that*

$$\phi_\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = h_\alpha(t), \quad \phi_\alpha \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_\alpha(t), \quad \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = w_\alpha(1) .$$

The elements of G have the following properties

1. $\alpha^\vee(t) = h_\alpha(t)$
2. For any $t \in \mathrm{T}$ we have $t x_\alpha(s) t^{-1} = x_\alpha(\alpha(t) s)$
3. For any $\alpha \in \Phi$ and any $\beta^\vee \in \Phi^\vee$ we have $\alpha(\beta^\vee(s)) = s^{<\alpha,\beta^\vee>}$
4. For any $w \in W$ and any $\alpha \in \Phi$ we have $w(\alpha^\vee) = (w\alpha)^\vee$ and $w \cdot x_\alpha = x_{w\alpha}$
5. $[x_\alpha(r), x_\alpha(s)] = \prod_{\substack{i,j>0 \\ i\alpha+j\beta\in\Phi}} x_{i\alpha+j\beta}\left(N_{i,j,\alpha,\beta} t^i s^j\right).$

Any $t \in \mathrm{T}(\mathrm{F})$ can be written uniquely as $t = \prod_{i=1}^{l} h_{\alpha_i}(t_i)$ with $t_i \in \mathrm{F}^\times$.

1.1.3 Structure of Parabolic Subgroups

A subgroup $\mathrm{B} \subset \mathrm{G}$ is called a *Borel subgroup* if it is a maximal Zariski closed solvable subgroup. All Borel subgroups of G are conjugate. A closed subgroup $\mathrm{P} \subset \mathrm{G}$ is called a *parabolic subgroup* if it contains a Borel subgroup.

Let G be the simply connected Chevalley group corresponding to a root system Φ as constructed in the previous section. Also, let T and U also be as defined there. Denote by $\mathrm{B} = \mathrm{T} \cdot \mathrm{U}$ (semi-direct product), which is a Borel subgroup of G.

Given a parabolic subgroup P, we have the following property

Proposition 1.3 (Levi decomposition)**.** *Suppose* P *is a connected algebraic group defined over* F*, then there exists a reductive subgroup* $\mathrm{M} \subset \mathrm{P}$ *such that* $\mathrm{P} = \mathrm{M}\, R_u(\mathrm{P})$ *(semi-direct product). The subgroup* M *is called a Levi component of* P*. This is an almost direct product, i.e. the intersection is finite.*

We have a classification of Borel subgroups as containing a fixed maximal torus T.

Theorem 1.4. *There is a one to one correspondence between Borel subgroups containing* T *and fundamental systems* Δ *of* Φ*. The correspondence is* $\Delta \leftrightarrow \mathrm{B}_\Delta$ *where*

$$\mathrm{B}_\Delta = \mathrm{T} \cdot \prod_{\alpha \in \Phi^+} U_\alpha \,,$$

where Φ^+ *is the set of positive roots in* Φ *determined by* Δ*.*

From now on we fix a fundamental system Δ and the corresponding Borel subgroup B. We have a classification of parabolic subgroups containing B.

Theorem 1.5. *There is a one to one correspondence between parabolic subgroups* P *containing* B_Δ *and subsets* $\Theta \subset \Delta$*. The correspondence is* $\Theta \leftrightarrow \mathrm{P}_\Theta$ *where*

$$\mathrm{P}_\Theta = \mathrm{G}(\Sigma_\Theta) \cdot \mathrm{T}_\Theta \cdot \mathrm{U}^+_\Theta \,,$$

with $\mathrm{G}(\Sigma_\Theta)$ *being the subgroup generated by* U_α *for* $\alpha \in \Sigma_\Theta = Span_{\mathbb{Z}}\{\Theta\} \cap \Phi$*,* $\mathrm{T}_\Theta = \left(\cap_{\alpha\in\Theta} \ker(\alpha)\right)^0$ *being the subtorus of* T *annihilated by* Θ *and* $U^+_\Theta = \prod_{\alpha \in \Phi_+ \setminus \Sigma^+_\Theta} U_\alpha$ *where* $\Sigma^+_\Theta = Span_{\mathbb{Z}}\{\Theta\} \cap \Phi_+$*.*

Remark 1.5. $M_\Theta = \mathrm{G}(\Sigma_\Theta) \cdot \mathrm{T}_\Theta$ is the Levi subgroup of P_Θ and $\mathrm{N}_\Theta = \mathrm{U}^+_\Theta$ is the unipotent radical of P_Θ.

Remark 1.6. The Borel subgroup B corresponds to the empty set in Δ. Also, if $\Theta_1 \subset \Theta_2 \subset \Delta$ then $\mathrm{P}_{\Theta_1} \subset \mathrm{P}_{\Theta_2}$. If $\Theta = \Delta \setminus \{\alpha\}$ for $\alpha \in \Delta$ then P_Θ is called a *maximal parabolic subgroup*.

Lemma 1.1 (Additional properties of parabolic subgroups)**.** *The following holds*

1. M_Θ *is the centralizer of* T_Θ *in* G*, i.e.* T_Θ *is the connected component of the center of* M_Θ*.*

2. $G(\Sigma_\Theta)$ *is the derived group of* M_Θ*.*

3. $T_\Theta \cap G(\Sigma_\Theta)$ *is finite.*

4. $G(\Sigma_\Theta)$ *is simply connected.*

1.2 Reductive Groups Over Local Fields

From now on we assume that F is a local non-Archimedean field of characteristic zero with $\mathcal{O}$ as its ring of integers and ϖ as a uniformizer of $\mathcal{O}$. We write G for G(F) where there is no confusion.

Let $K = G(\mathcal{O})$ be the standard maximal compact subgroup of G. Now we will describe two ways to decompose G as a product of certain subgroups. Both of the results can be found in the unpublished notes of Casselman [Cas74].

Proposition 1.4 (Iwasawa decomposition)**.** *For any parabolic subgroup* $P \subset G$ *we have* $G = PK$*.*

Remark 1.7. This is not a unique decomposition, i.e. one can find $p, p' \in P$ and $k, k' \in K$ such that $pk = p'k'$. Uniqueness is up to $P / (P \cap K)$.

We have

Proposition 1.5 (Cartan decomposition)**.** *The lattice* $\Lambda = T(F) / T(\mathcal{O})$ *is maximal in* T*. Denote*

$$A = \left\{ t \in \Lambda \middle| \, |\gamma(t)| \leq 1 \ \forall \gamma \in \Phi^+ \right\} .$$

There is a decomposition

$$G = KAK$$

such that any element $g \in G$ *can be uniquely written as* $g = kak'$ *with* $k, k' \in K, a \in A$*.*

1.3 The Chevalley Group G_2

1.3.1 The Root System

Let G = G_2 be the Chevalley group of the Lie algebra generated by the Dynkin diagram

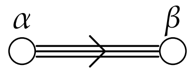


Figure 1.1: Dynkin diagram of G_2

where α is a short simple root and β is a long simple root. The group G_2 is simple and hence reductive and unimodular. It is also simply connected and adjoint, i.e. has trivial center. The set of the positive roots of G_2 is

$$\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$$

and the Cartan matrix of G_2 is

$$\begin{pmatrix} <\alpha, \alpha^\vee> & <\alpha, \beta^\vee> \\ <\beta, \alpha^\vee> & <\beta, \beta^\vee> \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

The fundamental weights of G_2 are

$$\omega_1 = 2\alpha+\beta, \quad \omega_2 = 3\alpha+2\beta\,,$$

where ω_1 is the highest weight of the standard 7-dimensional representation of G_2.

The Weyl group of G_2 is of order 12 and is generated by w_α and w_β and the relation $\left(w_\alpha w_\beta\right)^6 = \mathbb{1}$. Its action on the set of roots is given by

$$w_\alpha(\alpha) = \alpha - <\alpha, \alpha^\vee> \alpha = -\alpha$$
$$w_\alpha(\beta) = \beta - <\beta, \alpha^\vee> \alpha = 3\alpha+\beta$$
$$w_\beta(\alpha) = \alpha - <\alpha, \beta^\vee> \beta = \alpha+\beta$$
$$w_\beta(\beta) = \beta - <\beta, \beta^\vee> \beta = -\beta$$

and on the set of coroots it is given by

$$w_\alpha(\alpha^\vee) = \alpha^\vee - <\alpha, \alpha^\vee> \alpha^\vee = -\alpha^\vee$$
$$w_\alpha(\beta^\vee) = \beta^\vee - <\alpha, \beta^\vee> \alpha^\vee = \alpha^\vee + \beta^\vee$$
$$w_\beta(\alpha^\vee) = \alpha^\vee - <\beta, \alpha^\vee> \beta^\vee = \alpha^\vee + 3\beta^\vee$$
$$w_\beta(\beta^\vee) = \beta^\vee - <\beta, \beta^\vee> \beta^\vee = -\beta^\vee .$$

1.3.2 Maximal Parabolic Subgroups of G_2

First we will describe the parabolic subgroup P associated with the short root α. This subgroup is called the *Heisenberg parabolic*. The Levi factor M of P is isomorphic to GL(2) and generated by the simple root α since

$$\mathrm{Lie}(\mathrm{M}) = \mathrm{Lie}(\mathrm{T}) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} .$$

Let U be the unipotent radical P and

$$\mathrm{Lie}(\mathrm{U}) = \bigoplus_{\gamma \neq \alpha} \mathfrak{g}_\gamma .$$

The other maximal parabolic subgroup will be denoted by Q. We denote its Levi decomposition by $Q = LV$ where L is generated by β.

1.3.3 M Orbits of U

The Levi subgroup M acts on U by the adjoint action and hence it acts also on $\mathrm{Hom}(\mathrm{U}, \mathbb{G}_a)$. By [GGS02, Proposition 3.1] and [HMS98]:

Proposition 1.6. *There is a natural bijection between the* M*-orbits in* $\mathrm{Hom}(\mathrm{U}, \mathbb{G}_a)$ *and cubic algebras over* F.

Fix an additive character ψ of F. Composition with ψ gives a bijection between cubic algebras over F and complex characters on U(F).

For the rest of this paper we fix the character on U(F) corresponding to the **split** cubic algebra $\mathrm{F} \times \mathrm{F} \times \mathrm{F}$. This character, called the *split character*, is given by

$$\psi\left(x_{\beta}(r_1)\, x_{\alpha+\beta}(r_2)\, x_{2\alpha+\beta}(r_3)\, x_{3\alpha+\beta}(r_4)\, x_{3\alpha+2\beta}(r_5)\right) = \psi(r_2 + r_3)\ .$$

1.3.4 Embedding G_2 in $SO(7)$ and Chevalley Relations

From now on we shall identify G_2 with a subgroup of $SO(7)$ in the following way:

Let

$$\mathrm{SO}(7) = \left\{ g \in \mathrm{GL}(7) \,\middle|\, gJ^t g = J \right\}\ ,$$

where

$$J = \begin{pmatrix} & & & & & & 1 \\ & & & & & 1 & \\ & & & & 1 & & \\ & & & 1 & & & \\ & & 1 & & & & \\ & 1 & & & & & \\ 1 & & & & & & \end{pmatrix}.$$

The group G_2 can be embedded in SO (7) in a way that will be described in this section. Some crucial calculations in this thesis are done using this particular embedding.

We will now give explicit formulas for $x_\gamma(r) \in \mathrm{G}_2$ embedded in SO (7). We will do this only for $\gamma \in \Phi^+$since we have $x_{-\gamma}(r) = {}^t x_\gamma(r)$.

$$x_\alpha(r) = \begin{pmatrix} 1 & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -r & -\frac{r^2}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -r \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad x_\beta(r) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \mathrm{r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\mathrm{r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x_{\alpha+\beta}(r)=\begin{pmatrix}1&0&\mathrm{r}&0&0&0&0\\0&1&0&\mathrm{r}&0&\frac{-\mathrm{r}^2}{2}&0\\0&0&1&0&0&0&0\\0&0&0&1&0&-\mathrm{r}&0\\0&0&0&0&1&0&-\mathrm{r}\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix},\qquad x_{2\alpha+\beta}(r)=\begin{pmatrix}1&0&0&\mathrm{r}&0&0&\frac{-\mathrm{r}^2}{2}\\0&1&0&0&-\frac{\mathrm{r}}{2}&0&0\\0&0&1&0&0&\frac{\mathrm{r}}{2}&0\\0&0&0&1&0&0&-\mathrm{r}\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix}$$

$$x_{3\alpha+\beta}(r)=\begin{pmatrix}1&0&0&0&-\frac{\mathrm{r}}{2}&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&\frac{\mathrm{r}}{2}\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix},\qquad x_{3\alpha+2\beta}(r)=\begin{pmatrix}1&0&0&0&0&\frac{\mathrm{r}}{2}&0\\0&1&0&0&0&0&\frac{-\mathrm{r}}{2}\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix}.$$

The general toral element is of the form

$$(t_1,t_2):=\alpha^\vee(t_1)\cdot\beta^\vee(t_2)=\begin{pmatrix}t_1&&&&&&\\&t_1^{-1}t_2&&&&&\\&&t_1^2t_2&&&&\\&&&1&&&\\&&&&t_1^{-2}t_2&&\\&&&&&t_1t_2^{-1}&\\&&&&&&t_1^{-1}\end{pmatrix}.$$

We also denote

$$t_{n,m}:=(n\alpha^\vee+m\beta^\vee)(\varpi)=\begin{pmatrix}\varpi^n&&&&&&\\&\varpi^{m-n}&&&&&\\&&\varpi^{2n-m}&&&&\\&&&1&&&\\&&&&\varpi^{m-2n}&&\\&&&&&\varpi^{n-m}&\\&&&&&&{}^{-n}\end{pmatrix}.$$

We have

$$\alpha(t_{n,m}) = \varpi^{2n-m}, \quad \beta(t_{n,m}) = \varpi^{3n-2m} .$$

We also want to list some Chevalley relations in our group with respect to our choice of Chevalley basis. We have $\left[x_\gamma(r), x_\delta(s)\right] = 1$ with the exception of the following cases

$$\begin{aligned}
\left[x_\alpha(r), x_\beta(s)\right] &= x_{\alpha+\beta}(rs)\, x_{2\alpha+\beta}\left(r^2 s\right) x_{3\alpha+\beta}\left(r^3 s\right) x_{3\alpha+2\beta}\left(-2r^3 s^2\right) \\
\left[x_\alpha(r), x_{\alpha+\beta}(s)\right] &= x_{2\alpha+\beta}(2rs)\, x_{3\alpha+\beta}\left(3r^2 s\right) \\
\left[x_\alpha(r), x_{2\alpha+\beta}(s)\right] &= x_{3\alpha+\beta}(3rs) \\
\left[x_\beta(r), x_{3\alpha+\beta}(s)\right] &= x_{3\alpha+2\beta}(-rs) \\
\left[x_{\alpha+\beta}(r), x_{2\alpha+\beta}(s)\right] &= x_{3\alpha+2\beta}(3rs) \ .
\end{aligned}$$

1.3.5 Measuring the Double Cosets $\mathrm{K}\,t\,\mathrm{K}$

We fix the unique Haar measure on G such that $\int_{\mathrm{K}} dx = 1$ and think of it as a left Haar measure on the Borel subgroup $\mathrm{B} = \mathrm{T}\,\mathrm{N}$. The modulus character depends only on the values on the torus. For $t \in \mathrm{T}$ one has

$$\delta_{\mathrm{B}}(t) = \left|\prod_{\gamma>0} \gamma(t)\right| = \left|(10\alpha + 6\beta)(t)\right|$$

and thus for $t_{n,m} = (n\alpha^\vee + m\beta^\vee)(\varpi)$

$$\delta_{\mathrm{B}}(t_{n,m}) = q^{-2n-2m} .$$

The Haar measure on G factors in the following way:

$$\int_{\mathrm{G}} f(x)\, dx = \int_{\mathrm{B}} \int_{\mathrm{K}} f(bk)\, db dk = \int_{\mathrm{T}} \int_{\mathrm{N}} \int_{\mathrm{K}} f(tnk)\, \delta_{\mathrm{B}}(t)^{-1}\, dt\, dn\, dk .$$

We now describe the Cartan decomposition of G

Proposition 1.7. *Given the Lattice* $\Lambda = \{t_{n,m} | n, m \in \mathbb{Z}\}$, *then* $\mathrm{G} = \mathrm{K}\,\mathrm{A}\,\mathrm{K}$ *where*

$$\mathrm{A} = \{t_{n,m} \in \Lambda | 0 \le 3n \le 2m \le 4n\} .$$

For $t \in \mathrm{A}$ we shall need the measure of the double coset $\mathrm{K}\,t\,\mathrm{K}$. We have

Theorem 1.6 ([Mac71]). *For any $t \in \mathrm{A}$*

$$\operatorname{meas}(\mathrm{K}\,t\,\mathrm{K}) = \frac{\mu_{C_t}}{\mu_{\mathrm{G}}}\delta_{\mathrm{B}}^{-1}(t)\,, \tag{1.1}$$

where $C_t = \left\{g \in \mathrm{G} \middle| tgt^{-1} = g\right\}$ is the centralizer of t in G and when H is a split subgroup we have

$$\mu_{\mathrm{H}} = \frac{1}{\sum\limits_{w \in \mathrm{W_H}} q^{-l(w)}}\,. \tag{1.2}$$

In the case of G_2 we have

$$\mu_{\mathrm{G}} = \frac{1}{1 + 2q^{-1} + 2q^{-2} + 2q^{-3} + 2q^{-4} + 2q^{-5} + q^{-6}}\,.$$

In order to do this calculation in G_2 we are interested in the set A introduced in the previous section.

We denote $\mu_{n,m} = \mu(\mathrm{K}\,t_{n,m}\,\mathrm{K})$. If $t = e$, i.e. $n = m = 0$, one has $\mu_{0,0} = 1$ by the normalization of μ.

If $\alpha(t) = 1$, i.e. $2n = m > 0$, we have $C_{t_{n,2n}} \cong \mathrm{GL}(2)$. When $\beta(t) = 1$, i.e. $2m = 3n > 0$, we have $C_{t_{n,m}} \cong \mathrm{GL}(2)$. Note that these are two different Levi subgroups in G_2. In both cases

$$\mu_{n,m} = \frac{1 + 2q^{-1} + 2q^{-2} + 2q^{-3} + 2q^{-4} + 2q^{-5} + q^{-6}}{1 + q^{-1}}\delta_{\mathrm{B}}^{-1}(t_{n,m})\,.$$

Otherwise $0 < 3n < 2m < 4n$ and then $C_{t_{n,m}} = \mathrm{T} \cong k^\times \times k^\times$ which implies

$$\mu_{n,m} = \left(1 + 2q^{-1} + 2q^{-2} + 2q^{-3} + 2q^{-4} + 2q^{-5} + q^{-6}\right)\delta_{\mathrm{B}}^{-1}(t_{n,m})\,.$$

1.4 The Group *$Spin_8$*

In this section we will introduce another group that will be of use to us. Let $Spin_8$ be the simply connected Chevalley group of type D_4. The Dynkin diagram of this type is

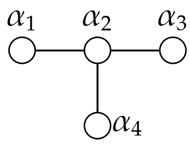


Figure 1.2: Dynkin diagram of D_4

It has the following set of positive roots

$$\Phi^+ = \left\{ \begin{matrix} (1000), (0100), (0010), (0001), (1100), (0110) \\ (0101), (1110), (1101), (0111), (1111), (1211) \end{matrix} \right\} ,$$

where $(n_1, n_2, n_3, n_4) = \sum_i n_i \alpha_i$. The Cartan matrix of $Spin_8$ is

$$\left(\left\langle \alpha_i, \alpha_j^\vee \right\rangle \right)_{i,j} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} .$$

For the generators of the Weyl group we use the notation $w_i = w_{\alpha_i}$. We will now list some Chevalley relations [Ste68] that will be helpful in the calculation.

$$[x_{(1100)}(u), x_{(0110)}(t)] = [x_{(1100)}(u), x_{(0101)}(t)] = [x_{(0101)}(u), x_{(0110)}(t)] = 1$$
$$[x_{-(1000)}(u), x_{(0110)}(t)] = [x_{-(1000)}(u), x_{(0011)}(t)] = 1$$
$$[x_{-(1000)}(u), x_{(1100)}(t)] = x_{(0100)}(-ut) \ .$$

1.5 Triality and the Embedding of G_2 in $Spin_8$

We consider the Dynkin diagram of D_4 as presented above and notice the fact that there is an action of S_3 (the symmetry group on 3 points) on the diagram. This action called *triality* induces a group of automorphisms on $Spin_8$ of order 6. We have

Lemma 1.2.

$$G_2 = (Spin_8)^{S_3} \ .$$

We write the map $\Psi : \mathrm{G}_2 \longrightarrow Spin_8$ explicitly on the elements x_γ with $\gamma \in \Phi^+$

$$\begin{aligned}
\Psi\left(x_\alpha(r)\right) &= x_{(1000)}(r)\, x_{(0010)}(r)\, x_{(0001)}(r) \\
\Psi\left(x_\beta(r)\right) &= x_{(0100)}(r) \\
\Psi\left(x_{\alpha+\beta}(r)\right) &= x_{(1100)}(r)\, x_{(0110)}(r)\, x_{(0101)}(r) \\
\Psi\left(x_{2\alpha+\beta}(r)\right) &= x_{(1110)}(r)\, x_{(0111)}(r)\, x_{(1101)}(r) \\
\Psi\left(x_{3\alpha+\beta}(r)\right) &= x_{(1111)}(r) \\
\Psi\left(x_{3\alpha+2\beta}(r)\right) &= x_{(1211)}(r) \; .
\end{aligned}$$

This map satisfies:

Lemma 1.3. Ψ *is an embedding of algebraic groups.*

Remark 1.8. We also note that there is a maximal parabolic $\mathrm{P_H}$ of H such that $\mathrm{P} = (\mathrm{P_H})^{S_3}$. This is the parabolic subgroup associated with $\{\alpha_1, \alpha_3, \alpha_4\}$.

Chapter 2

Representations and L-functions

2.1 Local Theory

Let F be a local non-Archimedean field. Also, let $\mathcal{O}$ be the ring of integers of F. Let ϖ be a uniformizer of $\mathcal{O}$ and q the cardinality of the residue field of F.

Let G be split reductive linear algebraic group defined over F. Let $\mathrm{K} = \mathrm{G}(\mathcal{O})$. Also, let B be a Borel subgroup of G with TN as a Levi decomposition of B. T is a maximal F-split torus. Let W be the Weyl group associated with T.

We fix the unique Haar measure μ on G such that $\mu(\mathrm{K}) = 1$.

For proofs of the results in this section see [Sat63] and the unpublished notes of W. Casselman [Cas74].

2.1.1 Local Representations

In this subsection we write G for G(F).

A *representation* of G is a pair (π, V) such that V is a complex vector space and $\pi : \mathrm{G} \longrightarrow \mathrm{GL}(V)$ is a group homomorphism. We will usually refer to the representation simply as π.

A representation π is *smooth* or *algebraic* if for any $v \in V$ the stabilizer of v in G is open. A representation π is *admissible* if it is smooth and for any open compact subgroup C of G the space V^C is finite-dimensional. An admissible representation π is *unramified* if $V^K \neq 0$. A non-zero vector $v_0 \in V^K$ is called a *spherical vector*.

Theorem 2.1. *An irreducible and smooth representation is also admissible.*

For any smooth representation π, the *contragredient representation* $\pi^\vee$ whose space is all the smooth linear functionals on V_π is defined to be

$$\forall g \in G, l \in \pi^\vee, v \in V_\pi : \ (\pi^\vee(g) l)(v) = l\left(\pi(g)^{-1} v\right) .$$

Given a parabolic subgroup $P \subset G$ with Levi subgroup M and unipotent subgroup N, and given a representation (σ, W) of M we may define the *normalized parabolic induction* of σ to G by

$$\mathrm{Ind}_P^G(\sigma) = \left\{ f : G \longrightarrow W \middle| \forall m \in M, n \in N, g \in G : f(mng) = \delta_P(m)^{1/2} \sigma(m) f(g) \right\} ,$$

with an action of G given by right translation. We recall that given a character χ of T and an element $w \in W$ we can define another character by

$$\forall t \in T : \quad ({}^w\chi)(t) = \chi({}^w t) \ .$$

Remark 2.1. A character of a group may be thought of as a one dimensional representation.

Theorem 2.2. 1. *For every irreducible unramified representation (π, V) the space V^K is one dimensional.*

2. *Given an unramified character χ of* T, *the representation* $\mathrm{Ind}_B^G(\chi)$ *admits a unique unramified irreducible subquotient π_χ.*

3. *The representations* $\mathrm{Ind}_B^G({}^w\chi)$ *for any $w \in$* W *all have the same Jordan-Hölder series only permuted. Also, for any such χ there exists $w \in$* W *such that* $\mathrm{Ind}_B^G({}^w\chi)$ *contains an irreducible unramified subrepresentation.*

4. *For any irreducible unramified representation π there exists a character χ of* T *such that π is a subrepresentation of* $\mathrm{Ind}_B^G(\chi)$.

5. *For any two characters* χ_1, χ_2 *of* T *we have*

$$\pi_{\chi_1} \cong \pi_{\chi_2} \iff \exists w \in \mathrm{W} : \ \chi_1 = {}^{w}\chi_2 \,.$$

Remark 2.2. Items 2, 3 and 4 of Theorem 2.2 allow us to think about irreducible unramified representations as spaces of functions.

2.1.2 The Langlands Dual Group

Fixing a split maximal torus T in G fixes a choice of root datum which is a quadruple $\Psi = (X^{*}(\mathrm{T}), \Phi, X_{*}(\mathrm{T}), \Phi^{\vee})$. We consider the lattice $\mathrm{T}(\mathrm{F}) / \mathrm{T}(\mathcal{O})$ and define the *dual torus* ${}^{L}\mathrm{T}$ of T to be

$${}^{L}\mathrm{T} = Spec\left(\mathbb{C}\left[\mathrm{T}(\mathrm{F}) / \mathrm{T}(\mathcal{O})\right]\right) \,.$$

We then can define

$${}^{L}\mathrm{T}(\mathbb{C}) = \mathrm{Hom}\left(\mathrm{T}(\mathrm{F}) / \mathrm{T}(\mathcal{O}), \mathbb{C}\right) \,.$$

This is a complex split torus.

Proposition 2.1. *There exists a complex split reductive linear algebraic group* ${}^{L}\mathrm{G}(\mathbb{C})$ *having* ${}^{L}\mathrm{T}(\mathbb{C})$ *as a maximal torus such that the root datum of* ${}^{L}\mathrm{T}(\mathbb{C})$ *with respect to* ${}^{L}\mathrm{G}(\mathbb{C})$ *is isomorphic to* $\Psi^{\vee} = (\mathrm{X}_{*}(\mathrm{T}), \Phi^{\vee}, X^{*}(\mathrm{T}), \Phi)$.

The group ${}^{L}\mathrm{G}(\mathbb{C})$ is called the *Langlands dual group* and is sometimes denoted by ${}^{L}\mathrm{G}$.

Remark 2.3. Note that this construction is equipped with natural bijections

$$X^{*}(\mathrm{T}) \leftrightarrow \mathrm{X}_{*}\left({}^{L}\mathrm{T}\right)$$
$$\mathrm{X}_{*}(\mathrm{T}) \leftrightarrow X^{*}\left({}^{L}\mathrm{T}\right)$$
$$\Phi \leftrightarrow \Phi^{\vee}$$
$$\Phi^{\vee} \leftrightarrow \Phi \,.$$

2.1.3 The Spherical Hecke Algebra

The *spherical Hecke algebra* $\mathcal{H} := \mathcal{H}(\mathrm{G},\mathrm{K}) = C_c(\mathrm{K}\setminus\mathrm{G}/\mathrm{K})$ is the algebra of locally constant, compactly supported and bi-K-invariant functions on G. The multiplication in this algebra is given by convolution, i.e.

$$(f_1 * f_2)(g) = \int_{\mathrm{G}} f_1(x) f_2\left(x^{-1}g\right) dx = \int_{\mathrm{G}} f_1\left(gx^{-1}\right) f_2(x)\, dx\,.$$

The equality is due to the fact that G is reductive and hence unimodular. The function $\mathbb{1}_{\mathrm{K}}$ is the unit of this algebra.

Remark 2.4. Note that $\mathcal{H}$ is not a C*-algebra. However, it can be equipped with a C*-norm and completed into a C*-algebra in the following manner. Due to the action of $\mathcal{H}$ on $L^2(K\setminus G/K)$ we may embed (as an algebra) $\mathcal{H}$ in $B(L^2(K\setminus G/K))$[1] and complete $\mathcal{H}$ with respect to the operator norm.

Given an admissible representation (π, V) of G we can turn it into a $\mathcal{H}$-module. The action of $f \in \mathcal{H}$ on $v \in V$ is given by

$$\pi(f)v = \int_{\mathrm{G}} f(g)\pi(g)v dg\,.$$

Since π is smooth and f is locally constant and compactly supported this integral is just a finite sum and thus this is well defined. For each $f \in \mathcal{H}$ we have $\pi(f)V \subset V^{\mathrm{K}}$ which is finite-dimensional by admissibility.

The map $(\pi, \mathrm{G}, V) \longrightarrow \left(\pi, \mathcal{H}, V^{\mathrm{K}}\right)$ is a map

admissible representations $\longrightarrow$ f. dim. $\mathcal{H}$-modules

irr. unramified representations $\longrightarrow$ 1 dim. $\mathcal{H}$-modules

Gelfand proved the following remarkable theorem concerning the spherical Hecke algebra

Theorem 2.3. *The algebra $\mathcal{H}$ is commutative.*

[1] The bounded operators on $L^2(K\setminus G/K)$

2.1.4 Satake Isomorphism and the Spectral Basis of $\mathcal{H}$

We recall that

$$^{L}\mathrm{T}(\mathbb{C}) = \mathrm{Hom}\left(\mathrm{T}(\mathrm{F})/\mathrm{T}(\mathcal{O}), \mathbb{C}\right) .$$

This is the group of unramified characters of T. By Theorem 2.2 this gives a bijection

$$\left\{\begin{matrix}\text{equivalence classes of}\\ \text{irreducible unramified}\\ \text{representations of } \mathrm{G}(\mathrm{F})\end{matrix}\right\} \leftrightarrow {}^{L}\mathrm{T}(\mathbb{C})/\mathrm{W} \leftrightarrow \left\{\begin{matrix}\text{semisimple conjugacy}\\ \text{classes of } {}^{L}\mathrm{G}\end{matrix}\right\}$$

For any unramified character χ we fix a representative $t_\chi \in {}^{L}\mathrm{T}(\mathbb{C})$ of the associated conjugacy class, which is called the *Satake parameter* of π_χ and is unique up to conjugacy by an element from W.

Remark 2.5. Note that $\mathrm{T}(\mathrm{F})$ is a reductive group and $\mathrm{T}(\mathcal{O})$ is a maximal compact subgroup. So the object $\mathcal{H}(\mathrm{T}(\mathrm{F}), \mathrm{T}(\mathcal{O}))$ is well defined as the space of locally constant compactly supported and bi-$\mathrm{T}(\mathcal{O})$-invariant functions on $\mathrm{T}(\mathrm{F})$.

We define the following $\mathbb{C}$-algebra homomorphisms

1. From the irreducible unramified representation (π_χ, V) we attain a 1-dimensional $\mathcal{H}$-module $\left(\pi_\chi, V^{\mathrm{K}}\right)$. Fix a spherical vector v_0. The action of $\mathcal{H}$ on v_0 define a character

$$\mathfrak{s}_{\mathrm{G},\chi} : \mathcal{H} \longrightarrow \mathbb{C} ,$$

given by

$$\int_{\mathrm{G}} f(x)\, \pi_\chi(x)\, v_0 dx = \mathfrak{s}_{\mathrm{G},\chi}(f)\, v_0 .$$

2. From the 1-dimensional representation $(\chi, \mathbb{C})$ of T we attain a 1-dimensional $\mathcal{H}(\mathrm{T}(\mathrm{F}), \mathrm{T}(\mathcal{O}))$-module. This yields a $\mathbb{C}$-algebra homomorphism

$$\mathfrak{s}_{\mathrm{T},\chi} : \mathcal{H}(\mathrm{T}(\mathrm{F}), \mathrm{T}(\mathcal{O})) \longrightarrow \mathbb{C}$$
$$g \mapsto \int_{\mathrm{T}} g(t)\, \chi(t)\, dt .$$

3. We have a natural evaluation map

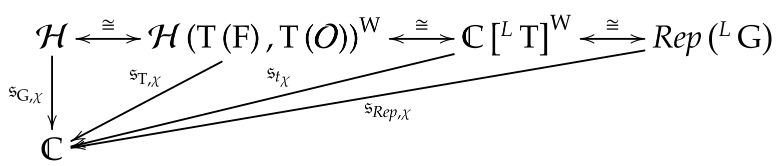
$$\mathfrak{s}_{t_\chi} : \mathbb{C}\left[{}^L\mathrm{T}\right]^{\mathrm{W}} \longrightarrow \mathbb{C}$$
$$h \mapsto h\left(t_\chi\right).$$

4. Any finite-dimensional representation of ${}^L\mathrm{G}$ has trace and thus we may form the following $\mathbb{C}$-algebra homomorphism

$$\mathfrak{s}_{Rep,\chi} : Rep\left({}^L\mathrm{G}\right) \longrightarrow \mathbb{C}$$
$$(\rho, V) \mapsto tr_V \rho\left(t_\chi\right) .$$

Now we can state the famous Satake isomorphism theorem:

Theorem 2.4 (Satake isomorphism)**.** *The following diagram of maps of* $\mathbb{C}$*-algebras commutes*



Moreover, the family of maps $\{\mathfrak{s}_{\mathrm{G},\chi} | \chi$ *unramified*$\}$ *supply a spectral decomposition of* $\mathcal{H}$*. In other words; for* $f \in \mathcal{H}$*, knowing the values* $\mathfrak{s}_{\mathrm{G},\chi}(f)$ *for all unramified* χ*, allows us to recover* f*.*

Remark 2.6. We have

$$\int_{\mathrm{G}} f(x)\, \pi_\chi(x)\, v_0 dx = \mathfrak{s}_{\mathrm{G},\chi}(f)\, v_0 .$$

Choosing a spherical vector $v_0 \in V^{\mathrm{K}} \subset \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi)$ such that $v_0(1) = 1$, we can realize $\mathfrak{s}_{\mathrm{G},\chi}$ by

$$\mathfrak{s}_{\mathrm{G},\chi}(f) = \left(\int_{\mathrm{G}} f(x)\, \pi_\chi(x)\, v_0 dx\right)(1) = \int_{\mathrm{G}} f(x)\, \pi_\chi(x)\, v_0(1)\, dx .$$

We can reformulate this as follows:

$$\begin{aligned}\mathfrak{s}_{G,\chi}(f) &= \int_G f(x)\,\pi_\chi(x)\,v_0(1)\,dx \\ &= \int_{NT}\int_K f(bk)\,\pi(bk)\,v_0(1)\,dbdk = \\ &= \int_{NT} f(b)\,v_0(b)\,db = \\ &= \int_T \chi(t)\,\delta_B^{\frac{1}{2}}(t)\int_N f(tn)\,dndt\ .\end{aligned}$$

So the isomorphism $\mathcal{H} \cong \mathcal{H}(\mathrm{T}(\mathrm{F}), \mathrm{T}(\mathcal{O}))$ is given by

$$\begin{aligned}\mathfrak{s} : \mathcal{H} &\longrightarrow \mathcal{H}(\mathrm{T}(\mathrm{F}), \mathrm{T}(\mathcal{O}))^W\ , \\ f &\mapsto \mathfrak{s}(f)\end{aligned}$$

where

$$\mathfrak{s}(f)(t) = \delta_B^{\frac{1}{2}}(t)\int_N f(tn)\,dn\ .$$

Corollary 2.1. *Fix a spherical vector $v_0 \in V_\chi$. For* ***any*** *$l \in \pi_\chi^\vee$ and any $f \in \mathcal{H}$ we have*

$$\int_G f(g)\,l(g\cdot v_0)\,dg = \mathfrak{s}_{G,\chi}(f)\,l(v_0)\ . \tag{2.1}$$

Proof. We have

$$\int_G f(g)\,l(g\cdot v_0)\,dg = l\left(\int_G f(g)\,\pi(g)\,v_0 dg\right) = l(\mathfrak{s}_{G,\chi}(f)\,v_0) = \mathfrak{s}_{G,\chi}(f)\,l(v_0)\ .$$

□

The spectral basis of $\mathcal{H}$ The Satake isomorphism gives rise to a bijection between finite-dimensional irreducible representations of ${}^L\mathrm{G}$ and a corresponding basis of $\mathcal{H}$. Each finite-dimensional irreducible representation of ${}^L\mathrm{G}$ is a highest weight module for a positive integral weight λ on ${}^L\mathrm{T}(\mathbb{C})$, whose corresponding function we denote as $A_\lambda \in \mathcal{H}$. By Theorem 2.4 we have

$$\int_G A_\lambda(x)\,\pi_\chi(x)\,v_0 dx = tr_{V_\lambda}(t_\chi)\,v_0 = \mathfrak{s}_{G,\chi}(A_\lambda)\,v_0\ . \tag{2.2}$$

2.1.5 Spherical Functions and Macdonald's Formula

Let (π, V) be an irreducible unramified representation. There exists an unramified character χ of T such that $\pi \hookrightarrow \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi)$.

We fix a spherical vector $v_0 \in V$, and let $v_0^\vee$ be the unique K-fixed functional in $\pi^\vee \hookrightarrow \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi^{-1})$ such that $\left\langle v_0, v_0^\vee \right\rangle = 1$. We define the *zonal spherical function* of π to be

$$\omega_\chi(g) = \left\langle \pi(g) v_0, v_0^\vee \right\rangle \ . \tag{2.3}$$

This is a bi-K-invariant function on G satisfying

$$\forall f \in \mathcal{H} \quad \int_{\mathrm{G}} f(x)\, \omega_\chi(x)\, dx \equiv \mathfrak{s}_{\mathrm{G},\chi}(f) \ . \tag{2.4}$$

In particular, for any positive integral weight λ we have

$$\int_{\mathrm{G}} A_\lambda(g)\, \omega_\chi(g)\, dg = tr_{V_\lambda}(t_\chi) = \mathfrak{s}_{\mathrm{G},\chi}(A_\lambda) \ . \tag{2.5}$$

The following theorem [Cas80, Theorem 4.2] gives an explicit formula for ω_χ. Let $\mathrm{G} = \mathrm{K\,A\,K}$ be the Cartan decomposition of G.

Theorem 2.5 (Macdonald's formula)**.** *Let χ be an unramified character such that $\langle \chi, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Phi$. For any $t \in \mathrm{A}$ we have*

$$\omega_\chi(t) = \mu_{\mathrm{G}} \delta_{\mathrm{B}}^{1/2}(t) \sum_{w \in \mathrm{W}} (\gamma_{{}^w\chi})\, {}^w\chi(t) \ ,$$

where

$$c_{\alpha,\chi} = \frac{1 - q^{-1}\chi(\alpha^\vee(\varpi))}{1 - \chi(\alpha^\vee(\varpi))}$$

and

$$\gamma_\chi = \prod_{\alpha \in \Phi^+} c_{\alpha,\chi} \ .$$

2.1.6 Fourier Transform

Let P be a parabolic subgroup of G with Levi decomposition $\mathrm{P} = \mathrm{M\,U}$. Also, fix a character ψ of U. Denote

$$C((\mathrm{U}, \psi) \setminus \mathrm{G} / \mathrm{K}) = \left\{ f : \mathrm{G} \longrightarrow \mathbb{C} \middle| \forall g \in \mathrm{G}, u \in \mathrm{U}, k \in \mathrm{K}\ f(ugk) = \psi(u) f(g) \right\} \ .$$

Remark 2.7. Due to Iwasawa decomposition, the functions in $C((\mathrm{U},\psi)\setminus \mathrm{G}/\mathrm{K})$ are determined by the values they attain on M.

The ψ-Fourier transform is a map

$$\begin{aligned} \mathcal{H} &\longrightarrow C((\mathrm{U},\psi)\setminus \mathrm{G}/\mathrm{K}) \ , \\ f &\mapsto f^{\psi} \end{aligned}$$

where

$$f^{\psi}(g) = \int_{\mathrm{U}} f(ug)\overline{\psi(u)}du \ .$$

Remark 2.8. This integral is always convergent since f is compactly supported. On the other hand, if $f^{\psi} \neq 0$ then f^{ψ} is not compactly supported but has compact support modulo U.

Proposition 2.2. *The Fourier transform is a map of right $\mathcal{H}$-modules, i.e. for any $f, g \in \mathcal{H}$ it holds $(f * g)^{\psi} = f^{\psi} * g$.*

Proof. Given $x \in \mathrm{G}$:

$$\begin{aligned} (f*g)^{\psi}(x) &= \int_{\mathrm{U}}\int_{\mathrm{G}} f(y)\, g\left(y^{-1}ux\right)\overline{\psi(u)}dy\, du = \\ = \{y = uz\} &= \int_{\mathrm{G}}\int_{\mathrm{U}} f(uz)\,\overline{\psi(u)}g\left(z^{-1}x\right) du\, dz = \\ &= \left(f^{\psi} * g\right)(x) \ . \end{aligned}$$

□

The following lemma is very useful when computing f^{ψ}.

Lemma 2.1. *Let $f \in C((\mathrm{U},\psi)\setminus \mathrm{G}/\mathrm{K})$. For $m \in \mathrm{M}$, if there exists an element $u \in \mathrm{U}\cap\mathrm{K}$ such that $\psi\left(mum^{-1}\right) \neq 1$ then $f(m) = 0$.*

Proof. Given $m \in \mathrm{M}$ and $u \in \mathrm{U}\cap\mathrm{K}$ such that $\psi\left(mum^{-1}\right) \neq 1$. Then we have

$$f(m) = f(mu) = f(u^{m}m) = \psi\left(mum^{-1}\right) f(m) \ .$$

Since $\psi\left(mum^{-1}\right) \neq 1$ we have $f(m) = 0$. □

2.2 Global Theory

Let k be a global field[2]. Denote by Π the set of places of k. For any v, k_v is a local field. For a finite place v it will be non-Archimedean and we denote by $\mathcal{O}_v$ the ring of integers of k_v with a uniformizer ϖ_v. Let the *adele* ring $\mathbb{A} = \mathbb{A}_k$ be the restricted product of all the local fields k_v with respect to $\mathcal{O}_v$. For non-Archimedean v let q_v denote the cardinality of the residue field of k_v.

Let G be a split reductive linear algebraic group defined over k. Thus G is defined over k_v for all $v \in \Pi$. For each finite v denote by $\mathrm{K}_v = \mathrm{G}(\mathcal{O}_v)$ the standard maximal compact subgroup of $\mathrm{G}(k_v)$. For archimedean v let K_v be a maximal compact subgroup of $\mathrm{G}(k_v)$.

We define $\mathrm{G}(\mathbb{A})$ to be the restricted product of the $\mathrm{G}(k_v)$ with respect to K_v. Also define by $\mathrm{K}(\mathbb{A}) = \prod_v \mathrm{K}_v$. Note that $\mathrm{K}(\mathbb{A})$ is a compact group.

We fix on $\mathrm{G}(k_v)$ the unique Haar measure μ_v with $\mu_v(\mathrm{K}_v) = 1$. Also denote $\mu = \prod_v \mu_v$.

The results in this section can be found in [CKM04].

2.2.1 Global Representations

A *global representation* is a pair (π, V) such that V is a $\mathbb{C}$ vector space and π is group homomorphism $\pi : \mathrm{G}(\mathbb{A}) \longrightarrow \mathrm{GL}(V)$. We call π irreducible if it does not contain a proper subrepresentation and smooth if the stabilizer of each vector $v \in V$ is an open subgroup of $\mathrm{G}(\mathbb{A})$.

The representation π is called admissible if for each class γ of smooth irreducible representations of K, the γ-isotypic component of V is finite-dimensional. The following fact is due to Flath [Fla79]

Theorem 2.6 (Tensor product)**.** *Given an irreducible admissible global representation π, there exists a unique set of irreducible admissible representations π_v*

[2]Global field=number field

of $\mathrm{G}(k_v)$ *such that* π_v *is unramified for almost all* v *and* $\pi = \otimes'_v \pi_v$ *(restricted tensor product).*

We are interested in a special class of global representations called cuspidal automorphic representations. The space of *automorphic forms* $\mathcal{A}(\mathrm{G}(k) \setminus \mathrm{G}(\mathbb{A}))$ is the space of all functions $f : \mathrm{G}(k) \setminus \mathrm{G}(\mathbb{A}) \longrightarrow \mathbb{C}$ such that:

1. f is locally constant

2. f is right $\mathrm{K}(\mathbb{A})$-finite, i.e. the right $\mathrm{K}(\mathbb{A})$-translations of $\mathrm{K}(\mathbb{A})$ span a finite dimension vector space

3. f is $Z(\mathfrak{g})$-finite where $Z(\mathfrak{g})$ is the center of the universal enveloping algebra of $\mathfrak{g}$

4. f is of moderate growth, i.e. given a norm on G, $f(g)$ is bounded by a polynomial in a certain norm $\|g\|$ on $\mathrm{G}(\mathbb{A})$.

This space fails to be a representation of $\mathrm{G}(\mathbb{A})$ as noted in [CKM04, Cogdell, lecture 3]. The reason for this is that $\mathrm{K}(\mathbb{A})$-finiteness is not preserved under the action of $\mathrm{G}(\mathbb{A})$, in particular this happen because of the action of the infinite places. One way to make it into a representation of $\mathrm{G}(\mathbb{A})$ is by enlarging. We define the space of *smooth automorphic forms* $\mathcal{A}^\infty(\mathrm{G}(k) \setminus \mathrm{G}(\mathbb{A}))$ is the space of all functions $f : \mathrm{G}(k) \setminus \mathrm{G}(\mathbb{A}) \longrightarrow \mathbb{C}$ such that:

1. f is locally constant

2. There is an open compact subgroup $L \subset \prod'_{v<\infty} \mathrm{K}_v$ such that $f(gl) = f(g)$ for all $l \in L$

3. There exists a Lie-ideal $\mathfrak{J} \subset Z(\mathfrak{g})$ of finite co-dimension such that $\mathfrak{J} f = 0$.

4. f is of uniform moderate growth, i.e. there exists a positive integer r such that for all differential operators $X \in \mathcal{U}(\mathfrak{g})$ (the universal enveloping algebra)
$$\left|Xf(g)\right| = C_X \left\|g\right\|^r .$$

By a theorem of Harish-Chandra this is a $\mathrm{G}(\mathbb{A})$ representation by right translations and also $\mathcal{A}(\mathrm{G}(k) \setminus \mathrm{G}(\mathbb{A})) \subset \mathcal{A}^\infty(\mathrm{G}(k) \setminus \mathrm{G}(\mathbb{A}))$.

In $\mathcal{A}^\infty(\mathrm{G}(k) \setminus \mathrm{G}(\mathbb{A}))$ we consider the subspace $\mathcal{A}_0^\infty(\mathrm{G}(k) \setminus \mathrm{G}(\mathbb{A}))$ of *cuspidal automorphic forms* consisting of all $f \in \mathcal{A}^\infty(\mathrm{G}(k) \setminus \mathrm{G}(\mathbb{A}))$ such that for all parabolic subgroups P with unipotent radical U the constant term vanishes
$$f_{\mathrm{U}}(g) := \int_{\mathrm{U}(k)\setminus\mathrm{U}(\mathbb{A}_k)} f(ug)\,du = 0 \quad \forall g \in \mathrm{G} .$$

This space decomposes into a direct sum of irreducible representations of $\mathrm{G}(\mathbb{A})$ called *cuspidal automorphic representation*. In this thesis we will be interested in cuspidal automorphic representations of G_2.

2.2.2 $\mathcal{L}$-Functions of Cuspidal Automorphic Representations

Let $\pi = \otimes\pi_v$ be an irreducible automorphic cuspidal representation of $G_{\mathbb{A}}$. There exists a finite set S of places such that π_v is unramified whenever $v \notin S$. By *Satake correspondence* we have a bijection:

$$\left\{\begin{array}{c}\text{equivalence classes of}\\ \text{irreducible unramified}\\ \text{representations of } \mathrm{G}_{k_v}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{semisimple conjugacy}\\ \text{classes of } {}^L\mathrm{G}\end{array}\right\}$$

By this bijection, for each unramified π_v there is a semi-simple element $t_{\pi_v} \in {}^L\mathrm{G}$, defined up to conjugacy. Let $\rho : {}^L\mathrm{G} \to GL(V)$ be a complex finite-dimensional representation. For each $v \notin S$, the local $\mathcal{L}$-function associated to π_v and ρ is defined to be

$$\mathcal{L}(s, \pi_v, \rho) = \frac{1}{\det(1 - q_v^{-s}\rho(t_{\pi_v}))} .$$

Define the partial global $\mathcal{L}$-function to be (at least formally)

$$\mathcal{L}^S(s,\pi,\rho) = \prod_{v \notin S} \mathcal{L}(s,\pi_v,\rho) \ . \tag{2.6}$$

Conjecture 2.1. *For the formal product defined in* (2.6) *we have*

1. *The infinite product converges for* $Re(s) >> 0$*.*

2. *The partial $\mathcal{L}$-function can be meromorphically continued for this product to the whole complex plane.*

3. $\mathcal{L}^S(\pi, s, \mathfrak{st})$ *admits a functional equation.*

This conjecture is proven for many cases via the Rankin-Selberg method which will be outlined in the following subsection. The Rankin-Selberg method relies on writing the $\mathcal{L}$-function as an integral whose meromorphic continuation is clear.

2.2.3 Rankin-Selberg Method

The Rankin-Selberg Method is a way to prove (2.1) through finding integral representations for the $\mathcal{L}$-function. We start by defining a global zeta-function $Z(\varphi, f_1, ..., f_n, s)$ where φ is a function in a given cuspidal representation π and $f_1, ..., f_n$ are functions independent of π. A typical example is

$$Z(\varphi, f, s) = \int_{\mathrm{G}(k)\backslash \mathrm{G}(\mathbb{A})} \varphi(h) E(f, h, s) dh \ , \tag{2.7}$$

where G is a subgroup of H, $E(f,h,s)$ is a normalized Eisenstein series on H. Once verifying the integral represents the $\mathcal{L}$-function the analytic properties of $\mathcal{L}^S(s,\pi,\rho)$ depend on those of the Eisenstein series.

Having introduced such an integral, the method advises us to proceed in the following two steps

1. **Factorizability** - given $\varphi, f_1, ..., f_n$ factorizable data where φ is a function corresponding to a pure tensor product $\otimes_v v_v$ in the underlying vector space of π. Show that

$$Z(\varphi, f_1, ..., f_n, s) = \prod_v Z_v(v_v, f_{v,1}, ..., f_{v,n}, s)$$

   for some local zeta-functions $Z_v(v_v, f_{v,1}, ..., f_{v,n}, s)$.

   Traditionally, in order to achieve step 1 in the R-S Method one rewrites the integral in step 1 as an integral over a purely adelic domain of a factorizable function. Classically, this process is called unfolding.

2. **Unramified Computation** - assume that π_v is unramified. Show that some unramified data $v_v, f_{v,1}, ..., f_{v,n}$ satisfies

$$Z_v(v_v, f_{v,1}, ..., f_{v,n}, s) = \mathcal{L}(\pi_v, s, \rho) .$$

There are a few classical examples of Rankin-Selberg integrals that use uniqueness of models of π.

- **The Jacquet-Langlands Method** [Bum97, Theorem 4.4.1] - for $G = \mathrm{GL}(n)$, ρ the standard representation and π an irreducible cuspidal and hence generic representation of G. Factorizability of the integral follows from the uniqueness of the *Whittaker model* of π.

- **The Godement-Jacquet Method** [JL70] for $G = \mathrm{GL}(n)$, ρ the standard representation and π irreducible and cuspidal. This time factorizability follows from the uniqueness of a G-invariant bilinear form on the underlying space of π. This method has been extended to classical groups and is called **the Doubling Method** [GPSR87].

In the late 1980's Piatetski-Shapiro and Rallis constructed in [PSR88] a Rankin-Selberg integral for the symplectic group $G = Sp_{2n}$ and the standard representation $\mathfrak{st}$ which does not unfold to a unique model. We follow their ideas in our work with the group G_2.

Chapter 3

Eisenstein Series of *Spin*$_8$

In this chapter, we take k to be a global field and $\mathrm{H} = Spin_8$ to be the simply-connected Chevalley group of type D_4. Let $\mathrm{P_H}$ be the Heisenberg parabolic subgroup of H with Levi decomposition $\mathrm{P_H} = \mathrm{M_H}\,\mathrm{U_H}$ as described in chapter 1.

The results in this chapter are taken from [GGJ02, Appendix].

We consider the induced representation from the character $\delta_{\mathrm{P_H}}^s$ with $s \in \mathbb{C}$. We denote

$$I_{\mathrm{P_H}}(s) := \mathrm{Ind}_{\mathrm{P_H}(\mathbb{A})}^{\mathrm{H}(\mathbb{A})}(\delta_{\mathrm{P_H}}^s) \ .$$

Note that we are using here unnormalized parabolic induction. An element $f \in I_{\mathrm{P_H}}(s)$ is a function $f : \mathrm{H}(\mathbb{A}) \longrightarrow \mathbb{C}$ such that f is right K-finite and

$$\forall p \in \mathrm{P_H}, h \in \mathrm{H} : f(ph) = \delta_{\mathrm{P_H}}^s(p)\, f(g) \ .$$

By Iwasawa decomposition we have $\mathrm{H}(\mathbb{A}) = \mathrm{P_H}(\mathbb{A}) \cdot \mathrm{K_H}(\mathbb{A})$, and so the values of an element in $I_{\mathrm{P_H}}(s)$ are determined by its restriction to $\mathrm{K_H}(\mathbb{A})$. Actually, there is a natural isomorphism of vector spaces between the restricted representation $\mathrm{Res}_{\mathrm{K_H}(\mathbb{A})}^{\mathrm{H}(\mathbb{A})}(I_{\mathrm{P_H}}(s))$ and

$$\left\{f : \mathrm{K} \longrightarrow \mathbb{C} \middle| \forall m \in \mathrm{M_H}(\mathbb{A}) \cap \mathrm{K_H}(\mathbb{A}) : f(mk) = f(k)\right\} \ .$$

Given a function f in this space, it can be extended to an element $f_s \in I_{\mathrm{P_H}}(s)$. The family $\left\{f_s \middle| s \in \mathbb{C}\right\}$ is called a *flat section* and the restriction of elements in the section to K is independent of s. Flat sections are also known as *standard sections*.

Given a flat section $f_s \in I_{P_H}(s)$ such that $f_s|_{K_H(\mathbb{A})} = f$, the *Eisenstein series* on $H(\mathbb{A})$ is defined to be

$$E(f,s,h) = \sum_{\gamma \in P_H(k)\backslash H(k)} f_s(\gamma h) \ .$$

Let us summarize its properties

Theorem 3.1. 1. *The Eisenstein series is left* $H(k)$*-invariant when it is defined.*

2. *There exists $c > 0$ such that the above sum is absolutely convergent for any f and g when $\mathfrak{Re}\, s > c$ and hence define a holomorphic function on the half plane $\mathfrak{Re}\, s > c$. This convergence is locally uniform in g.*

3. *For any f and $g \in H$ the function*

$$s \mapsto E(f,s,g)$$

admits a meromorphic continuation to all of $\mathbb{C}$.

4. *At a point s_0 where $E(f,s,g)$ is holomorphic for all f and g, the function $E(f,s_0,g)$ in g is an automorphic form on* H. *Also, the map*

$$f \mapsto E(f,s_0,-)$$

is a $H(\mathbb{A})$*-equivariant map of* $I_P(\sigma,s)$ *to* $\mathcal{A}(H(k) \backslash H(\mathbb{A}))$.

5. *For any s_0, there is a constant integer N such that*

$$\inf_{f,g}\{ord_{s=s_0} E(f,s,g)\} = -N$$

that is, the order of poles attained in s_0 as a function of f and g is bounded from below.

We are interested in the analytic behaviour of $E(f,s,g)$ at $s = \frac{4}{5}$. We have the following fact

Proposition 3.1 ([GGJ02], Proposition 9.1)**.** *For any standard section f, the Eisenstein series $E(f,s,g)$ has at most a double pole at $s=\frac{4}{5}$. The double pole is attained by the spherical section f^0.*

Also, the space

$$Span_{\mathbb{C}}\left(\left(s-\frac{4}{5}\right)^2 E(f,s,g)|_{s=\frac{4}{5}}\right) ,$$

is isomorphic to the minimal representation of $Spin_8$.

It is customary to normalize the Eisenstein series in the following way:

$$E^*(f,g,s) = \zeta(5s)\,\zeta(5s-1)^2\,\zeta(10s-4)E(f,g,s) \ . \tag{3.1}$$

Chapter 4

The Global Zeta Integral and Factorizability

Let k be a global field and let $\mathbb{A} = \mathbb{A}_k$ be the adele ring of k. Let G be the split simple exceptional linear algebraic group of type G_2 defined over k. Also, let H be the Chevalley group $Spin_8$ defined over k. For every place v let G_v denote $G(k_v)$ and K_v denote $G(\mathcal{O}_v)$.

We will now list the following subgroups of G and H which will be used in this chapter. For more details see chapter 1

- P denotes the Heisenberg parabolic subgroup of G associated with β. P has Levi decomposition $P = M\,U$.
- P_H denotes the Heisenberg parabolic subgroup of H such that $(P_H)^{S_3}$.

Denote by $E(f, s, g)$ the Eisenstein series associated with the unnormalized induced representation $\mathrm{Ind}_{P_H(\mathbb{A})}^{H(\mathbb{A})}(\delta_{P_H}^{s})$. Also denote by $E^*(f, s, g)$ the normalized Eisenstein series as introduced in chapter 3.

We denote by $V_{[n,m]}$ the finite-dimensional irreducible representation of $G_2(\mathbb{C})$ with $n\omega_1 + m\omega_2$ as its highest weight. Let $A_{[n,m]}$ be the corresponding function in $\mathcal{H}$ attached by Satake correspondence. We denote by $\mathfrak{st}$ the 7-dimensional representation $V_{[1,0]}$, which is called the *standard* representation of G_2.

4.1 The Zeta Integral

Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{G}(\mathbb{A})$. In this section we will introduce a zeta integral suggested by Dihua Jiang as a Rankin-Selberg integral for the standard $\mathcal{L}$-function of π.

We fix an additive character ψ of $\mathbb{A}/k$ and hence of $\mathrm{U}(\mathbb{A})/\mathrm{U}(k)$ as explained in chapter 1.

Let $L^\psi \in \mathrm{Hom}_{\mathrm{U}(\mathbb{A})}\left(V_\pi, \mathbb{C}_\psi\right)$ be defined by

$$L^\psi\left(\phi_\pi\right) = \int_{\mathrm{U}(k)\backslash\mathrm{U}(\mathbb{A})} \phi_\pi(u)\,\overline{\psi(u)}du\,.$$

This is the ψ-Fourier coefficient defined over V_π. We also denote

$$L^\psi\left(\phi_\pi\right)(g) = L^\psi\left(g\cdot\phi_\pi\right)$$

Remark 4.1. The space $\mathrm{Hom}_{\mathrm{U}(\mathbb{A})}\left(V_\pi, \mathbb{C}_\psi\right)$ is usually infinite dimensional and thus $L^\psi\left(\phi_\pi\right)$ is not necessarily factorizable even if ϕ_π is.

For a standard K-finite section f of $\mathrm{Ind}_{\mathrm{P_H}(\mathbb{A})}^{\mathrm{H}(\mathbb{A})}(\delta_{\mathrm{P_H}}^s)$ and a function $\phi_\pi \in \pi$ we form the following zeta integral,

$$\zeta\left(s, \phi_\pi, f\right) = \int_{\mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A})} E(f,s,g)\,\phi_\pi(g)\,dg\,. \tag{4.1}$$

Since ϕ_π is rapidly decreasing the integral is absolutely convergent and defines a meromorphic function on $\mathbb{C}$.

Also denote

$$\zeta^*\left(s, \phi_\pi, f\right) = \int_{\mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A})} E^*(f,s,g)\,\phi_\pi(g)\,dg\,.$$

Our goal is to prove the following:

Conjecture 4.1 (Main Conjecture). *Let π be an irreducible cuspidal automorphic representation of $\mathrm{G}_2(\mathbb{A})$ that has a non-vanishing split Fourier coefficient along the Heisenberg unipotent subgroup. For any factorizable data $\phi_\pi \in V_\pi$ and a section f_s there exists a finite set of places S such that*

$$\zeta^*\left(s, \phi_\pi, f\right) = \mathcal{L}^S(\pi, 5s-2, \mathfrak{st})\, d_S\left(s, f_S, \phi_{\pi_S}\right)\,, \tag{4.2}$$

where $d_S\left(s, f_S, \phi_{\pi_S}\right)$ *is a meromorphic function of* s.

Moreover, for any $s_0 \in \mathbb{C}$ *the data* $\phi_{\pi,v}$ *and* $f_{s,v}$ *for* $v \in S$ *can be chosen so that* $d_S\left(s, f_S, \phi_{\pi_S}\right)$ *is holomorphic and non-vanishing in a neighbourhood of* s_0.

Remark 4.2. The function F_s is factorizable if and only if f_s is. Let $F_v(g, f, s)$ be the local factor of F_s at v.

This is equivalent to

$$\zeta\left(s, \phi_\pi, f\right) = \frac{\mathcal{L}^S(\pi, 5s-2, \mathrm{st})}{j^S(s)} d_S\left(s, f_S, \phi_{\pi_S}\right) ,$$

where

$$j^S(s) = \zeta^S(5s)\,\zeta^S(5s-1)^2\,\zeta^S(10s-4)$$

is the normalizing factor introduced in chapter 3.

For such π the conjecture follows from the two theorems described below.

Theorem 4.1 (Unfolding of the Global Integral). *We have*

$$\zeta\left(s, \phi_\pi, f\right) = \int_{\mathrm{U}(\mathbb{A})\backslash \mathrm{G}(\mathbb{A})} L^\psi\left(\phi_\pi\right)(g)\, F_s(g)\, dg , \tag{4.3}$$

where

$$F_s(g) = \int_{\mathbb{A}} f_s\left(\mu\, x_{\alpha+\beta}(r)\, g\right) \psi(r)\, dr$$

with $\mu = w_2 w_3 x_{\alpha_1}(1)$

Remark 4.3. Any factorizable $\otimes_v f_{s,v} \in \mathrm{Ind}_{\mathrm{P_H}(\mathbb{A})}^{\mathrm{H}(\mathbb{A})}(\delta_{\mathrm{P_H}}^s)$ is unramified almost everywhere, i.e. for almost all v the vector $f_{s,v}$ is unramified. We fix $\otimes_v f_{s,v}$ such that $f_{s,v}$ is unramified and $f_{s,v}(1) = 1$ for almost all v.

Theorem 4.1 has been proven by Dihua Jiang. Since his result has never been published we present the computation in chapter 5.

Conjecture 4.2 (Unramified Computation). *Fix a finite* $v \in \Pi$ *such that* π_v, ψ_v *and* $f_{s,v}$ *are unramified. Fix* v_0 *to be the spherical vector in* π_v. *For* ***any*** $l \in \mathrm{Hom}_{\mathrm{U}(k_v)}\left(\pi_v, \mathbb{C}_\psi\right)$ *we have*

$$\int_{\mathrm{U}(k_v)\backslash \mathrm{G}(k_v)} F_v(g, s, f)\, l(g \cdot v_0)\, dg = \mathcal{L}(\pi_v, 5s-2, \mathrm{st})\, l(v_0) . \tag{4.4}$$

Let us derive Conjecture 4.1 from Theorem 4.1 and Conjecture 4.2.

Proof. Given pure tensor vectors $\phi_\pi = \otimes_\nu \phi_{\pi,\nu}$ and $f_s = \otimes_\nu f_{s,\nu}$ we fix S to be a finite set of places such that if $\nu \notin S$ then

- k_ν is non-Archimedean and the residual characteristic of k_ν is not 2 or 3.
- π_ν is unramified and $\phi_{\pi,\nu}$ is spherical.
- ψ_ν is unramified.
- $f_{s,\nu}$ is unramified and $f_{s,\nu}(1) = 1$.

For a finite set of places $\Omega \supseteq S$ we denote

$$\zeta_\Omega\left(s, \phi_\pi, f\right) = \int_{\mathrm{U}^\Omega \setminus \mathrm{G}^\Omega} L^\psi\left(\phi_\pi\right)(g)\, F(g, f, s)\, dg\ ,$$

where $\mathrm{G}^\Omega = \prod_{\nu\in\Omega} \mathrm{G}_{k_\nu} \times \prod_{\nu\notin\Omega} K_\nu$ and $\mathrm{U}^\Omega = \prod_{\nu\in\Omega} \mathrm{U}_{k_\nu} \times \prod_{\nu\notin\Omega} K_\nu$. We also denote

$$\mathcal{L}_{\Omega\setminus S}(\pi, s, \mathfrak{st}) = \prod_{\nu\in\Omega\setminus S} \mathcal{L}(\pi_\nu, s, \mathfrak{st})\ .$$

Remark 4.4. By definition of the partial $\mathcal{L}$-function

$$\mathcal{L}^S(\pi, s, \mathfrak{st}) = \lim_{\substack{\Omega \supseteq S \\ |\Omega| < \infty}} \mathcal{L}_{\Omega\setminus S}(\pi, s, \mathfrak{st})\ . \tag{4.5}$$

On the other hand,

$$\zeta\left(s, \phi_\pi, f\right) = \lim_{\substack{\Omega \supseteq S \\ |\Omega| < \infty}} \zeta_\Omega\left(s, \phi_\pi, f\right) \tag{4.6}$$

Remark 4.5. Since all datum for $\nu \notin S$ is spherical[1] we may write

$$\int_{\mathrm{U}^\Omega \setminus \mathrm{G}^\Omega} L^\psi\left(\phi_\pi\right)(g)\, F(g, f, s)\, dg = \int_{\mathrm{U}_\Omega \setminus \mathrm{G}_\Omega} L^\psi\left(\phi_\pi\right)(g)\, F(g, f, s)\, dg\ ,$$

[1] Haar measures are normalized such that $\int_{K_\nu} dg_\nu = 1$

where $\mathrm{G}_\Omega = \prod_{v\in\Omega} \mathrm{G}_{k_v}$ and $\mathrm{U}_\Omega = \prod_{v\in\Omega} \mathrm{U}_{k_v}$ with the embedding $\mathrm{G}_\Omega \hookrightarrow \mathrm{G}^\Omega$ given by $g \mapsto g \otimes \otimes_{v\notin\Omega} 1$.

Lemma 4.1 (Inductive Process). *For every finite set $\Omega \supset S$ and any place $v \notin \Omega$ we have*

$$\zeta_{\Omega\cup\{v\}}(s) = \frac{\mathcal{L}\left(\pi_v, 5s-2, \mathfrak{st}\right)}{j_v\left(s\right)}\, \zeta_\Omega(s)\ .$$

Proof of Lemma 4.1. Let Λ_g be the functional on π_v defined as follows

$$\Lambda_g\left(v\right) = L^\psi\left(g\cdot\left(\otimes_{\mu\neq v}\phi_{\pi,\mu}\otimes v\right)\right) \quad \forall v \in V_{\pi,v}\ .$$

For $g_v \in \mathrm{G}_v$ it satisfies

$$\Lambda_g\left(g_v\cdot v_v^0\right) = L^\psi\left(\phi_\pi\right)\left(gg_v\right)\ ,$$

so that $\Lambda_g \in \mathrm{Hom}_{\mathrm{U}_v}\left(\pi_v, \mathbb{C}_{\psi_v}\right)$.

For any place $v \notin \Omega$ we have

$$\begin{aligned}
\zeta_{\Omega\cup\{v\}}(s) &= \int_{\mathrm{U}_\Omega\setminus\mathrm{G}_\Omega} F(g,f,s)\int_{\mathrm{U}_v\setminus\mathrm{G}_v} L^\psi\left(\phi_\pi\right)(gg_v)F_v(g_v,f,s)dg_v\, dg = \\
&= \int_{\mathrm{U}_\Omega\setminus\mathrm{G}_\Omega} F(g,f,s)\int_{\mathrm{U}_v\setminus\mathrm{G}_v} \Lambda_g(g_v\cdot v_v^0)F_v(g_v,f,s)dg_v\, dg = \\
&= \frac{\mathcal{L}\left(\pi_v, 5s-2, \mathfrak{st}\right)}{j_v\left(s\right)}\int_{\mathrm{U}_\Omega\setminus\mathrm{G}_\Omega} \Lambda_g(v_v^0)F(g,f,s)dg = \\
&= \frac{\mathcal{L}\left(\pi_v, 5s-2, \mathfrak{st}\right)}{j_v\left(s\right)}\int_{\mathrm{U}_\Omega\setminus\mathrm{G}_\Omega} L^\psi\left(\phi_\pi\right)(g)\, F(g,f,s)dg\ .
\end{aligned}$$

□

By induction on $|\Omega|$ it follows that for any finite set of places $\Omega \supseteq S$

$$\zeta_\Omega(s) = \frac{\mathcal{L}_{\Omega\setminus S}\left(\pi, 5s-2\, \mathfrak{st}\right)}{j_{\Omega\setminus S}\left(s\right)}\cdot\int_{\mathrm{U}_S\setminus\mathrm{G}_S} L^\psi\left(\phi_\pi\right)\left(g_S\right)F_S\left(g_S,s\right)dg_S\ .$$

Plugging this into eq. (4.5) and eq. (4.6) yields

$$\begin{aligned}\zeta\left(s,\phi_\pi,f\right) &= \lim_{\substack{\Omega\supseteq S\\|\Omega|<\infty}} \int_{U(\mathbb{A})_\Omega\backslash G(\mathbb{A})_\Omega} L^\psi\left(\phi_\pi\right)(g)\,F_s(g)\,dg = \\ &= \lim_{\substack{\Omega\supseteq S\\|\Omega|<\infty}} \prod_{v\in\Omega\backslash S} \mathcal{L}\left(5s-2,\pi_v,\mathfrak{st}\right) \int_{U(\mathbb{A})_S\backslash G(\mathbb{A})_S} L^\psi\left(\phi_\pi\right)(g)\,F_s(g)\,dg = \\ &= \mathcal{L}^S\left(5s-2,\pi,\mathfrak{st}\right)\int_{U(\mathbb{A})_S\backslash G(\mathbb{A})_S} L^\psi\left(\phi_\pi\right)(g)\,F_s(g)\,dg\ .\end{aligned}$$

Hence the equality in Conjecture 4.1. We plan to prove the meromorphic continuation of $d_S\left(s,f_s,\phi_{\pi_S}\right)$ in the near future. □

4.2 Unramified Computation

It remains to prove Conjecture 4.2. Our strategy will be outlined in the following section. Since this section is purely local we omit the subscript ν. We let $\mathrm{F}=k_\nu$ with a ring of integers $\mathcal{O}$ and a uniformizer ϖ. Let G denote G (F).

Let π be an irreducible unramified representation of G (F) with Satake parameter t_χ. Fix a spherical vector $v_0\in V_\pi$.

The basic fact we rely on is

Proposition 4.1. *There exists a function $\Delta_s\in\mathcal{H}\left(\mathrm{G},\mathrm{K}\right)[[q^{-s}]]$ such that for **any** $l\in\pi_\nu^\vee$ we have*

$$\int_{\mathrm{G}}\Delta_s(g)\,l\left(g\cdot v_0\right)dg=\mathcal{L}\left(\pi,s,\mathfrak{st}\right)l\left(v_0\right)\ .$$

Proof. We have the *Poincare identity*

$$\begin{aligned}\mathcal{L}(\pi,s,\mathfrak{st}) &= \frac{1}{\det\left(1-q^{-s}\,\mathfrak{st}(t_\chi)\right)} = \prod_{i=1}^{7}\left(1-q^{-s}\,\mathfrak{st}\left(t_\chi\right)_{ii}\right)^{-1} = \\ &= \prod_{i=1}^{7}\sum_{n=0}^{\infty}\left(q^{-s}\,\mathfrak{st}\left(t_\chi\right)_{ii}\right)^n = \sum_{n=0}^{\infty} tr_{Sym^n}\left(\mathfrak{st}\left(t_\chi\right)\right)q^{-ns}\ .\end{aligned}$$

By [RS89] the decomposition of symmetric powers of the standard representation is given by

$$Sym^n(\mathfrak{st}) = \begin{cases} \oplus_{k=0}^{\frac{n}{2}} V_{[2k,0]}, & n \equiv 0 \pmod 2 \\ \oplus_{k=0}^{\frac{n-1}{2}} V_{[2k+1,0]}, & n \equiv 1 \pmod 2 \end{cases}$$

and therefore

$$\mathcal{L}(\pi, s, \mathfrak{st}) = \frac{\sum_{n=0}^{\infty} tr_{V_{[n,0]}}(\mathfrak{st}(t_\chi))\, q^{-ns}}{1 - q^{-2s}} .$$

Due to Theorem 2.4 our assertion is proved by choosing

$$\Delta_s(g) = \zeta(2s)\left(\sum_{n=0}^{\infty} A_{[n,0]}(g)\, q^{-ns}\right) .$$

□

Fix $l \in \mathrm{Hom}_{\mathrm{U(F)}}\left(\pi, \mathbb{C}_\psi\right)$. By bi-K-invariance we have

$$\begin{aligned} l(v_0)\,\mathcal{L}(\pi, s, \mathfrak{st}) &= \int_{\mathrm{G}} \Delta_s(g)\, l(g \cdot v_0)\, dg = \\ &= \int_{\mathrm{U}\backslash\mathrm{G}} \int_{\mathrm{U}} \Delta_s(ug)\, l(ug \cdot v_0)\, du\, dg = \\ &= \int_{\mathrm{U}\backslash\mathrm{G}} l(ug \cdot v_0)\left(\int_{\mathrm{U}} \Delta_s(ug)\, \psi(u)\, du\right) dg = \\ &= \int_{\mathrm{U}\backslash\mathrm{G}} \Delta_s^{\psi}(g)\, l(g \cdot v_0)\, dg . \end{aligned}$$

Thus, in order to prove

$$\int_{\mathrm{U}\backslash\mathrm{G}} F_\nu(g, s, f)\, l(g \cdot v_0)\, dg = \mathcal{L}(\pi_\nu, 5s-2, \mathfrak{st})\, l(v_0) ,$$

we need to prove

$$\int_{\mathrm{U}\backslash\mathrm{G}} \frac{\Delta_{5s-2}^{\psi}(g)}{j(s)} l(g \cdot v_0)\, dg = \int_{\mathrm{U}\backslash\mathrm{G}} F_\nu(g, s, f)\, l(g \cdot v_0)\, dg .$$

It will suffice to show that

$$\frac{\Delta_{5s-2}^{\psi}}{j(s)} = F_s .$$

Remark 4.6. The analogues equality in [PSR88] was proven by direct computation of Δ_s. Unfortunately in our work on G_2 a nice formula for Δ_s was inaccessible. In order to bypass this problem we introduce the following simple function.

Let K A K be the Cartan decomposition as discussed in chapter 1. Let $D_s \in \mathcal{H}[[q^{-s}]]$ be the function defined on A as follows

$$D_s(t) = |\omega_1(t)|_F^{5s+1} \qquad \forall t \in A$$

The function D_s has the following property:

Proposition 4.2. *There exists a function $P_s \in \mathcal{H}[q^{-s}]$ such that for any unramified character χ*

$$\mathfrak{s}_{G,\chi}(D_s) = \mathfrak{s}_{G,\chi}(\Delta_{5s-2})\, \mathfrak{s}_{G,\chi}(P_s)$$

and in particular

$$D_s = \Delta_{5s-2} * P_s\ .$$

Proof. For any $l \in \pi_\chi^\vee$ we have

$$\int_G f(g)\, l(gv_0)\, dg = \mathfrak{s}_{G,\chi}(f)\, l(v_0)\ .$$

Take a spherical vector $v_0{}^\vee \in \pi_\chi^\vee$ such that $\langle v_0, v_0{}^\vee\rangle = 1$ and define $l(v) = \langle v, v_0{}^\vee\rangle$ so that $\omega_\chi(g) = l(g \cdot v_0)$. The spherical function satisfies $\omega_\chi(1) = 1$.

For $\chi(t_{n,m}) = \eta^n \nu^m$ with $\eta, \nu \in \mathbb{C}$ the image of the Satake parameter of π_χ under $\mathfrak{st}$ is

$$\mathfrak{st}(t_\chi) = \begin{pmatrix} \eta\nu^2 &&&&&& \\ & \eta\nu &&&&& \\ && \nu &&&& \\ &&& 1 &&& \\ &&&& \nu^{-1} && \\ &&&&& \eta^{-1}\nu^{-1} & \\ &&&&&& \eta^{-1}\nu^{-2} \end{pmatrix}.$$

By Macdonald's formula, Theorem 2.5 and the formula measuring double cosets, Proposition 1.7, we get

$$\mathfrak{s}_{G,\chi}(D) = \int_G D_s(g)\,\omega_\chi(g)\,dg = \sum_{0\le 3n\le 2m\le 4n} D_s(t_{n,m})\,\omega_\chi(t_{n,m})\,\mu_{n,m} =$$
$$= \sum_{w\in W}\gamma_{{}^w\chi}\sum_{0\le 3n\le 2m\le 4n} D_s(t_{n,m})\,\mu_{n,m}\mu_G\delta^{1/2}(t_{n,m})\,{}^w\chi(t_{n,m}) = \sum_{w\in W}\gamma_{{}^w\chi} f(s,{}^w\chi)\ ,$$

where

$$\gamma_\chi = \frac{1-q^{-1}\eta^{-1}}{1-\eta^{-1}}\frac{1-q^{-1}\nu^{-1}}{1-\nu^{-1}}\frac{1-q^{-1}\eta^{-1}\nu^{-1}}{1-\eta^{-1}\nu^{-1}}\frac{1-q^{-1}\eta^{-1}\nu^{-2}}{1-\eta^{-1}\nu^{-2}}\frac{1-q^{-1}\eta^{-1}\nu^{-3}}{1-\eta^{-1}\nu^{-3}}\frac{1-q^{-1}\eta^{-2}\nu^{-3}}{1-\eta^{-2}\nu^{-3}}$$

and

$$\begin{aligned} f(s,\chi) = {} & \frac{1}{1+2q^{-1}+2q^{-2}+2q^{-3}+2q^{-4}+2q^{-5}+q^{-6}} + \\ & + \frac{1}{1+q^{-1}}\sum_{p=1}^{\infty} D_s\left(t_{2p,3p}\right)\delta_P^{-1/2}\left(t_{2p,3p}\right)\chi\left(t_{2p,3p}\right) + \\ & + \frac{1}{1+q^{-1}}\sum_{n=1}^{\infty} D_s(t_{n,2n})\,\delta_P^{-1/2}(t_{n,2n})\,\chi(t_{n,2n}) + \\ & + \sum_{p=2}^{\infty}\sum_{m=3p+1}^{4p-1} D_s(t_{2p,m})\delta_P^{-1/2}(t_{2p,m})\chi(t_{2p,m}) + \\ & + \sum_{p=1}^{\infty}\sum_{m=3p+2}^{4p+1} D_s\left(t_{2p+1,m}\right)\delta_P^{-1/2}\left(t_{2p+1,m}\right)\chi\left(t_{2p+1,m}\right)\ . \end{aligned}$$

After simplifying the expression we have

$$\int_G D_s(g)\,\omega_\chi(g)\,dg = \frac{P_1\left(q^{-(5s-2)}\right) - P_2\left(q^{-(5s-2)}\right)\mathrm{tr}_{V_{[1,0]}}\,t_\chi}{\zeta(5s-1)\,\zeta(5s+1)\,\zeta(5s-2)}\,\mathcal{L}(\pi, 5s-2, \mathfrak{st})\ , \tag{4.7}$$

here

$$P_1(z) := \frac{z^4}{q^2} + \left(\frac{1}{q^2}+\frac{1}{q}\right)z^3 + \frac{z^2}{q} + \left(\frac{1}{q}+1\right)z + 1, \quad P_2(z) := \frac{z^2}{q}\ .$$

By definition we have

$$\mathfrak{s}_{G,\chi}(A_{[0,0]}) = 1$$
$$\mathfrak{s}_{G,\chi}(A_{[1,0]}) = \mathrm{tr}_{V_{[1,0]}}(t_\chi)\ ,$$

hence $\mathfrak{s}_{G,\chi}(D_s) = \mathfrak{s}_{G,\chi}(\Delta_s)\,\mathfrak{s}_{G,\chi}(P_s)$ is proven by choosing

$$P_s = \frac{P_1\left(q^{-(5s-2)}\right) A_{[0,0]} - P_2\left(q^{-(5s-2)}\right) A_{[1,0]}}{\zeta(5s-1)\,\zeta(5s+1)\,\zeta(5s-2)} \,. \tag{4.8}$$

Since the Satake transform is a $\mathbb{C}$-algebras map it holds $\mathfrak{s}_{G,\chi}(D_s) = \mathfrak{s}_{G,\chi}(\Delta_s * P_s)$ for any unramified χ. Since this is a spectral decomposition of $\mathcal{H}$ we have $D_s = \Delta_{5s-2} * P_s$. □

Since the Fourier transform is a map of $\mathcal{H}$-modules we have

Corollary 4.1.

$$D_s^{\psi} = \Delta_{5s-2}^{\psi} * P_s \,. \tag{4.9}$$

On the other hand, convolution with P_s is an invertible map as shown in the following proposition

Proposition 4.3. *For $f \in C((\mathrm{U},\psi)\setminus \mathrm{G}/\mathrm{K})$ if for $\mathfrak{Re}\, s >> 0$ it holds $f * P_s \equiv 0$ then $f \equiv 0$.*

Proof. this is equivalent to showing that $x := A_{[0,0]} - \frac{q^{3-10s}}{P_1(q^{-s})} A_{[1,0]}$ is invertible for $\mathfrak{Re}\, s >> 0$.

Since $\mathcal{H}[q^{-s}]$ is embedded inside a C*-algebra we need only show that $\left\| \frac{q^{3-10s}}{P_1(q^{-s})} A_{[1,0]} \right\| < 1$. We will show this for large enough $\mathfrak{Re}\, s$.

We have

$$\left\| \frac{q^{3-10s}}{P_1(q^{-s})} A_{[1,0]} \right\| = \left| \frac{q^{3-10s}}{P_1(q^{-s})} \right| \left\| A_{[1,0]} \right\| \leq \frac{\left| q^{3-10s} \right|}{\left| 1 - \left| p(q^{-s}) \right| \right|} \left\| A_{[1,0]} \right\| \,,$$

where $p := P_1 - 1$ is a polynomial such that $p(0) = 0$. As $\mathfrak{Re}\, s \to \infty$ we have $p(q^{-s}) \to 0$. There exists s_0 such that for any $\mathfrak{Re}\, s \geq s_0$ we have $\frac{\left|q^{3-10s}\right|}{\left|1-\left|p(q^{-s})\right|\right|} < \left\| A_{[1,0]} \right\|$ and thus x is invertible for such s. □

By Corollary 4.1 and Proposition 4.3 the proof of Conjecture 4.2 is reduced to proving the following equality

$$D_s^{\psi} = j(s)\, F_s * P_s \,. \tag{4.10}$$

The equality eq. (4.10) will be checked directly for the unit element of G_2. The calculation of D_s^{ψ} is carried in chapter 6 for toral elements. The calculation of F_s is done in chapter 7. The computation of the convolution $F_s * P_s$ is also performed in chapter 7 at the identity element.

Chapter 5

Unfolding of the Integral

Let k be a global field and let $\mathbb{A} = \mathbb{A}_k$ be the adele ring of k. Let G be the split simple exceptional linear algebraic group of type G_2 defined over k. Also, let H be the Chevalley group $Spin_8$ defined over k.

We will now list the following subgroups of G and H which are going to be used in this chapter. For more details see chapter 1

- P denotes the Heisenberg parabolic subgroup G associated with β. P has Levi decomposition $P = M\,U$.
- Q denotes the other maximal parabolic subgroup in G associated with α. Q has Levi decomposition $Q = L\,V$.
- Y denotes the unipotent radical of L.
- $Z = [U, U]$ which is a 1-dimensional space and is actually the center of U. Z is generated by $3\alpha + 2\beta$
- $R = [V, V]$ which is an abelian unipotent subgroup of Q generated by the positive roots $2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta$.
- P_H denotes the Heisenberg parabolic subgroup of H such that $(P_H)^{S_3}$.

Denote by $E(f, s, g)$ the Eisenstein series associated to the unnormalized induced representation $\mathrm{Ind}_{P_H(\mathbb{A})}^{H(\mathbb{A})}(\delta_{P_H}^s)$.

In this chapter we prove the unfolding of the global integral as stated in Conjecture 4.1:

Theorem 5.1. *We have*

$$\zeta\left(s,\phi_\pi\right)=\int_{\mathrm{U}(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})} L^\psi\left(\phi_\pi\right)(g)\,F_s(g)\,dg\,, \tag{5.1}$$

where

$$\begin{aligned} L^\psi\left(\phi_\pi\right)(g) &= \int_{\mathrm{U}(k)\backslash\mathrm{U}(\mathbb{A})} \phi_\pi(ug)\,\psi\overline{(u)}du \\ F_s(g) &= \int_{\mathbb{A}} f_s\left(\mu\, x_{\alpha+\beta}(r)\, g\right)\psi(r)\,dr \end{aligned}$$

The proof follows unpublished notes made by Dihua Jiang. The decomposition of $\mathrm{P_H}\backslash\mathrm{H}$ into G orbits is due to [Jia98].

Lemma 5.1 ([Jia98] Lemma 2.1)**.** *The flag variety* $\mathrm{P_H}\backslash\mathrm{H}$ *decomposes into* G*-orbits as follows:*

$$\mathrm{H}=[\mathrm{P_H\,G}]\cup[\mathrm{P_H}\,w_2w_1\,\mathrm{G}]\cup[\mathrm{P_H}\,w_2w_3\,\mathrm{G}]\cup[\mathrm{P_H}\,w_2w_4\,\mathrm{G}]\cup[\mathrm{P_H}\,w_2w_3x_{-\alpha_1}(1)\,\mathrm{G}]\ .$$

The stabilizer in G *of each orbit* $\mathrm{P}\,\mu\,\mathrm{G}$ *equals*

$$\mathrm{G}^\mu=\begin{cases} \mathrm{P}, & \mu=1\\ \mathrm{L}\cdot R, & \mu=w_2w_1,w_2w_3,w_2w_4\\ \left\{(3\alpha+2\beta)^\vee, x_\beta, \left[x_{\alpha+\beta}x_{2\alpha+\beta}\right]^\Delta, x_{3\alpha+\beta}, x_{3\alpha+2\beta}\right\}, & \mu=w_2w_3x_{-\alpha_1}(1)\end{cases}\ ,$$

where $\left[x_{\alpha+\beta}x_{2\alpha+\beta}\right]^\Delta$ *is the diagonal embedding* $r\mapsto x_{\alpha+\beta}(r)\,x_{2\alpha+\beta}(r)$

We consider the following global zeta integral

$$\zeta\left(s,\phi_\pi,f\right)=\int_{\mathrm{G}(k)\backslash\mathrm{G}(\mathbb{A})} E(g;s,f_s)\,\phi_\pi(g)\,dg\,. \tag{5.2}$$

Using the definition of the Eisenstein series we have

$$\begin{aligned}\zeta\left(s,\phi_\pi,f\right) &= \int_{\mathrm{G}(k)\backslash\mathrm{G}(\mathbb{A})} \sum_{\gamma\in\mathrm{P_H}(k)\backslash\mathrm{H}(k)} f_s(\gamma g)\,\phi_\pi(g)\,dg = \\ &= \int_{\mathrm{G}(k)\backslash\mathrm{G}(\mathbb{A})} \sum_{\mu\in\mathrm{P_H}(k)\backslash\mathrm{H}(k)/\,\mathrm{G}(k)} \sum_{\gamma\in\mathrm{G}^\mu(k)\backslash\mathrm{G}(k)} f_s(\mu\gamma g)\,\phi_\pi(g)\,dg = \\ &= \sum_{\mu\in\mathrm{P_H}(k)\backslash\mathrm{H}(k)/\,\mathrm{G}(k)} \int_{\mathrm{G}^\mu(k)\backslash\mathrm{G}(\mathbb{A})} f_s(\mu g)\,\phi_\pi(g)\,dg = \\ &= \sum_{\mu\in\mathrm{P_H}(k)\backslash\mathrm{H}(k)/\,\mathrm{G}(k)} \zeta^\mu\left(s,\phi_\pi,f\right)\,,\end{aligned}$$

where

$$\zeta^\mu\left(s,\phi_\pi,f\right) = \int_{\mathrm{G}^\mu(k)\backslash\mathrm{G}(\mathbb{A})} f_s(\mu g)\,\phi_\pi(g)\,dg\,.$$

The first step toward proving Theorem 4.1 is to show that all but one of the $\zeta^\mu\left(s,\phi_\pi,f\right)$ vanish.

Lemma 5.2. *For* $\mu = 1, w_2w_1, w_2w_3, w_2w_4$

$$\zeta^\mu\left(s,\phi_\pi\right) \equiv 0\,.$$

Proof. We can separate this into two cases.

Case I: Assume $\mu = 1$, then

$$\begin{aligned}\zeta^\mu\left(s,\phi_\pi\right) &= \int_{\mathrm{P}(k)\backslash\mathrm{G}(\mathbb{A})} f_s(g)\,\phi_\pi(g)\,dg = \\ &= \int_{\mathrm{M}(k)\,\mathrm{U}(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})} \int_{\mathrm{U}(k)\backslash\mathrm{U}(\mathbb{A})} f_s(ug)\,\phi_\pi(ug)\,du\,dg \\ &= \int_{\mathrm{M}(k)\,\mathrm{U}(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})} f_s(g)\left(\int_{\mathrm{U}(k)\backslash\mathrm{U}(\mathbb{A})} \phi_\pi(ug)\,du\right) dg = 0\,.\end{aligned}$$

The inner integral is identically 0 since π is cuspidal.

Case II: Assume $\mu = w_2w_1, w_2w_3, w_2w_4$, then up to conjugation by an element in $\mathrm{G}(k)$ we have $\mathrm{G}^\mu = \mathrm{L}\cdot R$ and then

$$\begin{aligned}\zeta^\mu\left(s,\phi_\pi\right) &= \int_{[\mathrm{L}\cdot R](k)\backslash\mathrm{G}(\mathbb{A})} f_s(\mu g)\,\phi_\pi(g)\,dg = \\ &= \int_{\mathrm{L}(k)\cdot R(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})} f_s(\mu g) \int_{R(k)\backslash R(\mathbb{A})} \phi_\pi(zg)\,dz\,dg\,.\end{aligned}$$

By [RS89, Theorem 5] we have

$$\int_{R(k)\backslash R(\mathbb{A})} \phi_\pi(zg)\,dz = \int_{R(k)\backslash R(\mathbb{A})} \left(g\cdot\phi_\pi\right)(z)\,dz = \sum_{\nu\in Y(k)\backslash \mathrm{L}(k)} W^{\psi}_{\phi_\pi}(\nu g)\ ,$$

where $W^{\psi}_{\phi_\pi}(g)$ is the Whittaker-Fourier coefficient of ϕ_π with respect to the additive character ψ. Since R is abelian, this is simply the Fourier decomposition of ϕ_π. If π is not generic[1] then

$$\zeta^{\mu}\left(s,\phi_\pi\right) = 0\ .$$

On the other hand, if π is generic then we have

$$\begin{aligned}\zeta^{\mu}\left(s,\phi_\pi\right) &= \int_{\mathrm{L}(k)\cdot R(\mathbb{A})\backslash \mathrm{G}(\mathbb{A})} f_s(\mu g) \sum_{\nu\in Y(k)\backslash L(k)} W^{\psi}_{\phi_\pi}(\nu g)\,dg = \\ &= \int_{Y(k)\cdot R(\mathbb{A})\backslash \mathrm{G}(\mathbb{A})} f_s(\mu g)\, W^{\psi}_{\phi_\pi}(g)\,dg = \\ &= \int_{[Y\cdot R](\mathbb{A})\backslash \mathrm{G}(\mathbb{A})} f_s(\mu g) \int_{Y(k)\backslash Y(\mathbb{A})} W^{\psi}_{\phi_\pi}(ug)\,du\,dg\end{aligned}$$

since $\mu\,\mathrm{U_H}\,\mu^{-1}\subset \mathrm{U_H}$ and f_s is left-$\mathrm{U_H}$ invariant. On the other hand, standard computation shows that

$$\zeta^{\mu}\left(s,\phi_\pi\right) = \int_{[Y\cdot R](\mathbb{A})\backslash \mathrm{G}(\mathbb{A})} f_s(\mu g)\, W^{\psi}_{\phi_\pi}(g)\,dg \int_{Y(k)\backslash Y(\mathbb{A})} \psi_0(u)\,du$$

and since

$$\int_{Y(k)\backslash Y(\mathbb{A})} \psi_0(u)\,du = 0\ ,$$

we get

$$\zeta^{\mu}\left(s,\phi_\pi\right) \equiv 0\ .$$

□

By Lemma 5.1 we are left with integrating on the open orbit $\mathrm{P}\,w_2w_3x_{-\alpha_1}(1)\,\mathrm{G}$ and get

$$\zeta\left(s,\phi_\pi\right) = \zeta^{\mu}\left(s,\phi_\pi\right)\ ,$$

[1]Since being generic means that one of these coefficients is not zero

where $\mu = w_2 w_3 x_{-\alpha_1}(1)$. The stabilizer of this orbit is $G^{\mu} = T_1 U_1$ where

$$\begin{aligned} T_1 &:= \left\{(3\alpha+2\beta)^{\vee}\right\} \\ U_1 &:= \left\{x_{\beta}, \left[x_{\alpha+\beta}x_{2\alpha+\beta}\right]^{\Delta}, x_{3\alpha+\beta}, x_{3\alpha+2\beta}\right\} . \end{aligned}$$

We now turn to prove Theorem 4.1.

Proof of Theorem 4.1. We have already shown that

$$\zeta\left(s,\phi_{\pi}\right) = \int_{[T_1 U_1](F)\backslash G(\mathbb{A})} f_s(\mu g)\, \phi_{\pi}(g)\, dg$$

with $\mu = w_2 w_3 x_{-\alpha_1}(1)$. Therefore

$$\begin{aligned} \zeta\left(s,\phi_{\pi}\right) &= \int_{[T_1 U_1](F)\backslash G(\mathbb{A})} f_s(\mu g)\, \phi_{\pi}(g)\, dg = \\ &= \int_{T_1(F)\, U_1(\mathbb{A})\backslash G(\mathbb{A})} \int_{U_1(k)\backslash U_1(\mathbb{A})} f_s(\mu u g)\, \phi_{\pi}(ug)\, du\, dg = \\ &= \int_{T_1(F)\, U_1(\mathbb{A})\backslash G(\mathbb{A})} f_s(\mu g) \int_{U_1(k)\backslash U_1(\mathbb{A})} \phi_{\pi}(ug)\, du\, dg , \end{aligned}$$

since $\mu\, U_1\, \mu^{-1} \in U_H$ and f_s is left U_H invariant. For any $u \in U_1$ we denote

$$u = u(x,r,-r,y,z) = x_{\beta}(x)\, x_{\alpha+\beta}(r)\, x_{2\alpha+\beta}(-r)\, x_{3\alpha+\beta}(y)\, x_{3\alpha+2\beta}(z) .$$

Fourier decomposition over the compact abelian group $k \setminus \mathbb{A}$ yields

$$\phi_{\pi}\left(x_{\alpha+\beta}(l)\, g\right) = \sum_{\psi \in \widehat{k\backslash\mathbb{A}}} \psi(l)\, C_{\psi} ,$$

where

$$C_{\psi} = \int_{k\backslash\mathbb{A}} \phi_{\pi}(u(x,r+s,-r,y,z)\, g)\, \overline{\psi(s)} ds .$$

We remember that $\widehat{k \setminus \mathbb{A}} = k$ and set a nontrivial character ψ; any other character is of the form

$$\psi_a(x) = \psi(ax)$$

for $a \in k$. If we put $l = 0$ in the Fourier decomposition we get

$$\phi_{\pi}(g) = \sum_{a\in k} \int_{k\backslash\mathbb{A}} \phi_{\pi}\left(x_{\alpha+\beta}(s)\, g\right) \overline{\psi(as)} ds .$$

Plugging this into $\int_{U_1(k)\backslash U_1(\mathbb{A})} \phi_\pi(ug)\,du$ we get

$$\begin{aligned}
&\int_{U_1(k)\backslash U_1(\mathbb{A})} \phi_\pi(ug)\,du = \\
&= \int_{U_1(k)\backslash U_1(\mathbb{A})} \sum_{a\in k} \int_{k\backslash\mathbb{A}} \phi_\pi(u(x, r+s, -r, y, z)\,g)\,\overline{\psi(as)\,ds}\,du\ .
\end{aligned}$$

Changing variables to $r' = r + s$ gives us

$$\begin{aligned}
&\int_{U_1(k)\backslash U_1(\mathbb{A})} \phi_\pi(ug)\,du = \\
&= \sum_{a\in k} \int_{U_1(k)\backslash U_1(\mathbb{A})} \int_{k\backslash\mathbb{A}} \phi_\pi(u(x, r', -r, y, z)\,g)\,\overline{\psi(ar')\psi(ar)} dr'\,du = \\
&= \sum_{a\in k} \int_{U_1(k)\backslash U_1(\mathbb{A})} \int_{k\backslash\mathbb{A}} \phi_\pi\left(u(x, s, r, y, z)\left(a^2, a\right) g\right) \overline{\psi(r+s)} dr\,du = \\
&= \sum_{a\in k^\times} \int_{U_1(k)\backslash U_1(\mathbb{A})} \int_{k\backslash\mathbb{A}} \phi_\pi\left(u(x, s, r, y, z)\left(a^2, a\right) g\right) \overline{\psi(r+s)} dr\,du + \\
&+ \int_{U_1(k)\backslash U_1(\mathbb{A})} \int_{k\backslash\mathbb{A}} \phi_\pi\left(u(x, s, r, y, z)\left(a^2, a\right) g\right) \overline{\psi(r+s)} dr\,du = \\
&= \sum_{a\in k^\times} \int_{U(k)\backslash U(\mathbb{A})} \phi_\pi\left(u(x, s, r, y, z)\left(a^2, a\right) g\right) \overline{\psi(r+s)} du + \\
&+ \int_{U(k)\backslash U(\mathbb{A})} \phi_\pi\left(u(x, s, r, y, z)\left(a^2, a\right) g\right) \overline{\psi(r+s)} du\ .
\end{aligned}$$

Since π is cuspidal the latter integral vanishes. Remembering that $T_1 = \{(a^2, a) \big| a \in k^\times\}$ we get

$$\begin{aligned}
\zeta\left(s, \phi_\pi\right) &= \int_{T_1(F)\,U_1(\mathbb{A})\backslash G(\mathbb{A})} f_s(\mu g) \cdot \\
&\quad \cdot \sum_{a\in k^\times} \int_{U(k)\backslash U(\mathbb{A})} \phi_\pi\left(u(x, s, r, y, z)\left(a^2, a\right) g\right) \overline{\psi(r+s)} du\,dg = \\
&= \int_{U_1(\mathbb{A})\backslash G(\mathbb{A})} f_s(\mu g) \int_{U(k)\backslash U(\mathbb{A})} \phi_\pi(ug)\,\overline{\psi(u)} du\,dg\ ,
\end{aligned}$$

where the definition of $\psi(u)$ for $u \in U$ is obvious. Finally, using another change of variables $u' = ux_{\alpha+\beta}(r)$ we get

$$\begin{aligned}
\zeta\left(s,\phi_\pi\right) &= \int_{U_1(\mathbb{A})\backslash G(\mathbb{A})} f_s(\mu g) \int_{U(k)\backslash U(\mathbb{A})} \phi_\pi(ug)\,\overline{\psi(u)}du\,dg = \\
&= \int_{U(\mathbb{A})\backslash G(\mathbb{A})} \int_{\mathbb{A}} \int_{U(k)\backslash U(\mathbb{A})} \phi_\pi\left(ux_{\alpha+\beta}(r)\,g\right)\overline{\psi(u)} f_s\left(\mu\, x_{\alpha+\beta}(r)\,g\right) du\,dr\,dg = \\
&= \int_{U(\mathbb{A})\backslash G(\mathbb{A})} \int_{\mathbb{A}} L^\psi\left(\phi_\pi\right)\left(x_{\alpha+\beta}(r)\,g\right) f_s\left(\mu\, x_{\alpha+\beta}(r)\,g\right)\psi(r)\,dr\,dg \\
&= \int_{U(\mathbb{A})\backslash G(\mathbb{A})} L^\psi\left(\phi_\pi\right)(g) \int_{\mathbb{A}} f_s\left(\mu\, x_{\alpha+\beta}(r)\,g\right)\psi(r)\,dr\,dg\,.
\end{aligned}$$

□

Chapter 6

The Fourier Transform of D

Let F be a local field of residual characteristic not equal to either 2 or 3. Let ν be the valuation of F, $\mathcal{O}$ its ring of integers and ϖ a uniformizer of $\mathcal{O}$.

Let $\mathrm{G} = \mathrm{G}_2(\mathrm{F})$, $\mathrm{K} = \mathrm{G}(\mathcal{O})$ its usual maximal compact subgroup, B the Borel subgroup and $\mathrm{P} = \mathrm{M}\,\mathrm{U}$ the Heisenberg parabolic subgroup.

We set an additive character ψ of F having conductor $\mathcal{O}$ [1]. As in 1, ψ defines a character on U.

6.1 Helpful Computational Lemmas

First, we would like to recall a fact about the Haar measure of F:

Lemma 6.1. *Making a change of variables $x = \alpha + \beta y$ is translated to $dx = |\beta|_{\mathrm{F}}\, dy$ at the level of the Haar measure.*

Next, we make Lemma 2.1 explicit for G_2.

Proposition 6.1. *The values of $f \in C((\mathrm{U},\psi)\setminus \mathrm{G}/\mathrm{K})$ are determined be the values the function attains at the points $\alpha^\vee(t_1)\,\beta^\vee(t_2)\,x_\alpha(d)$ and it is zero unless the following inequalities are satisfied:*

[1] This means that $\psi|_{\mathcal{O}} \equiv 1$ and $\psi\left(\varpi^{-1}\right) \neq 1$

1. *If $d \in \mathcal{O}$ then*
$$t_1, \frac{t_2}{t_1} \in \mathcal{O}\,.$$

2. *If $d \notin \mathcal{O}$ we have*
$$t_1, \frac{t_2}{t_1}, dt_1 \in \mathcal{O}$$
and also either $d^2\frac{t_1^3}{t_2} \in \mathcal{O}$ or $d^2t_1, \frac{dt_2}{t_1} \in \mathcal{O}$.

Proof. Since $f \in C\left(\left(U,\psi\right) \setminus G\,/\,K\right)$, its values are determined by the values it attains in M. Each element in M is of the form $g = \alpha^\vee(t_1)\beta^\vee(t_2)x_\alpha(d)$. M has Levi decomposition $m = tnk$ where t is toral, n is unipotent and $k \in M(\mathcal{O}) \subset K$. By right-K-invariance we need only consider elements of the form $g = \alpha^\vee(t_1)\beta^\vee(t_2)x_\alpha(d)$ with either $d \notin \mathcal{O}$ or $d = 0$ (representing $d \in \mathcal{O}$).

By Lemma 2.1, for $m \in M$, if $f(m) \neq 0$ then $\psi\left(mum^{-1}\right) = 1$ for all $u \in U \cap K$. On the other hand, every element $u \in U$ is of the form

$$u = x_\beta(r_1)\,x_{\alpha+\beta}(r_2)\,x_{2\alpha+\beta}(r_3)\,x_{3\alpha+\beta}(r_4)\,x_{3\alpha+2\beta}(r_5)$$

and $u \in U \cap K$ if and only if $r_1, r_2, r_3, r_4, r_5 \in \mathcal{O}$.

Since ψ is an additive character we may assume u is any one of $x_\beta(r_1), x_{\alpha+\beta}(r_2), x_{2\alpha+\beta}(r_3), x_{3\alpha+\beta}(r_4), x_{3\alpha+2\beta}(r_5)$ with $r_i \in \mathcal{O}$ instead of considering a product of such elements. Also, since $\psi(u) = \psi(r_2 + r_3)$ we only need to consider $x_\beta(r_1), x_{\alpha+\beta}(r_2), x_{2\alpha+\beta}(r_3)$.

Case 1: Assume $u = x_\beta(r_1)$ then

$$\begin{aligned}
&f(g)\,f\left(gx_\beta(r_1)\right) = \\
&= f\left(x_{\alpha+\beta}\left(\frac{dt_2}{2t_1}r_1\right)x_{2\alpha+\beta}\left(\frac{d^2t_1}{4}r_1\right)x_{3\alpha+\beta}\left(\frac{d^3t_1^3}{8t_2}r_1\right)x_{3\alpha+2\beta}\left(\frac{-d^3t_2}{2}r_1^2\right)x_\beta\left(\frac{t_2^3}{t_1^3}r_1\right)g\right) = \\
&= \psi\left(\left(\frac{dt_2}{2t_1} + \frac{d^2t_1}{4}\right)r_1\right)f(g)
\end{aligned}$$

and thus $\frac{dt_2}{2t_1} + \frac{d^2t_1}{4} \in \mathcal{O}$.

Case 2: Assume $u = x_{\alpha+\beta}(r_2)$ then

$$\begin{aligned} & f(g) f\left(g x_{\alpha+\beta}(r_2)\right) = \\ & = f\left(x_{2\alpha+\beta}(t_1 d r_2)\, x_{3\alpha+\beta}\left(\frac{3d^2 t_1^3}{4t_2} r_2\right) x_{\alpha+\beta}\left(\frac{t_2}{t_1} r_2\right) g\right) = \\ & = \psi\left(\left(t_1 d + \frac{t_2}{t_1}\right) r_2\right) f(g) \end{aligned}$$

and thus $t_1 d + \frac{t_2}{t_1} \in \mathcal{O}$.

Case 3: Assume $u = x_{2\alpha+\beta}(r_3)$ then

$$f(g) = f\left(g x_{2\alpha+\beta}(r_3)\right) = f\left(x_{3\alpha+\beta}\left(\frac{3dt_1^3}{2t_2} r_3\right) x_{2\alpha+\beta}(t_1 r_3)\, g\right) = \psi(t_1 r_3) f(g)$$

and thus $t_1 \in \mathcal{O}$.

The cases $u = x_{3\alpha+\beta}(r_4)$ and $u = x_{3\alpha+2\beta}(r_5)$ do not contribute information.

If we assume that $d \in \mathcal{O}$ or rather $d = 0$, as explained above, this reduces to only $t_1, \frac{t_2}{t_1} \in \mathcal{O}$.

On the other hand, if $d \notin \mathcal{O}$ then we have $t_1, dt_1 + \frac{t_2}{t_1}, \frac{dt_2}{2t_1} + \frac{d^2 t_1}{4} \in \mathcal{O}$. The latter two yield $dt_1, \frac{t_2}{t_1} \in \mathcal{O}$ and we claim that either $d^2 \frac{t_1^3}{t_2} \in \mathcal{O}$ or $d^2 t_1, \frac{dt_2}{t_1} \in \mathcal{O}$. Assuming that the second does not hold, we must have $\left|\frac{dt_2}{t_1}\right|_{\mathrm{F}} = \left|d^2 t_1\right|_{\mathrm{F}}$ and thus $\left|\frac{t_1^2}{t_2}\right|_{\mathrm{F}} = \left|\frac{1}{d}\right|_{\mathrm{F}} < 1$ and thus also $d^2 \frac{t_1^3}{t_2} \in \mathcal{O}$. □

We repeatedly use the following two facts. Note that we prove them for ψ but they are also true for $\overline{\psi}$.

Lemma 6.2. *For* $j \in \mathbb{Z}$

$$\int_{\varpi^j \mathcal{O}^\times} \psi(x)\, dx = \begin{cases} 0, & j < -1 \\ -1, & j = -1 \\ q^{-j}\left(1 - q^{-1}\right), & j \geq 0 \end{cases}.$$

Proof. The result for $j \geq 0$ follows from the fact that ψ is trivial on $\mathcal{O}$. For $j < -1$, by a change of variables:

$$\int_{\varpi^j \mathcal{O}^\times} \psi(x)\, dx = \int_{\varpi^j \mathcal{O}^\times} \psi\left(x' + \varpi^{-1}\right) dx' = \psi\left(\varpi^{-1}\right) \int_{\varpi^j \mathcal{O}^\times} \psi(x)\, dx .$$

By assumption $\psi(\varpi) \neq 1$ and thus $\int_{\varpi^j \mathcal{O}^\times} \psi(x)\, dx = 0$. A similar argument shows that

$$\int_{\mathrm{F}} \psi(x)\, dx = 0$$

and hence

$$0 = \int_{\mathcal{O}} \psi(x)\, dx + \sum_{j=1}^{\infty} \int_{\varpi^j \mathcal{O}^\times} \psi(x)\, dx = 1 + \int_{\varpi^{-1}\mathcal{O}^\times} \psi(x)\, dx\,.$$

From this the result follows. □

The following lemma is a volume calculation which we will encounter often during the calculation of D^ψ.

Lemma 6.3. *For $a, b \in \mathrm{N}$ with $a \leq b$ let $A \subset \mathrm{F}^2$ be the set described by the inequalities*

$$|x|_{\mathrm{F}} \leq q^a, \quad |y|_{\mathrm{F}}, |xy|_{\mathrm{F}} \leq q^b\,.$$

The volume of A equals $q^b + aq^b\left(1 - q^{-1}\right)$.

Proof.

$$\begin{aligned} Vol(A) = & \int_{\mathcal{O}} dx \int_{\varpi^{-b}\mathcal{O}} dy + \sum_{j=1}^{a} \int_{\varpi^{-j}\mathcal{O}^\times} dx \int_{\varpi^{j-b}\mathcal{O}^\times} = \\ = & q^b + \sum_{j=1}^{a} q^j \left(1 - q^{-1}\right) q^{b-j} = \\ = & q^b + aq^b\left(1 - q^{-1}\right)\,. \end{aligned}$$

□

6.2 The Fourier Transform of D

Recall the definition of D_s given in chapter 4. Let K A K be the Cartan decomposition of G. $D_s \in \mathcal{H}\left[\left[q^{-s}\right]\right]$ is the bi-K-invariant function defined on A by

$$D_s(t) = |\omega_1(t)|_F^{5s+1}\,.$$

In this section we compute the ψ-Fourier coefficient of D_s for toral elements. In chapter 7 we use only the value of D_s^{ψ} at e but we this computation is needed for the complete proof of Conjecture 4.2. The computation for non-toral elements is not more complicated but it is more technically involved and is omitted in this thesis.

Proposition 6.2 (Fourier Transform)**.** *We have*

$$D^{\psi}\left(t_{n,m}\right) = \begin{cases} \frac{q^{-n(5s+1)}\left(1+2q^{1-5s}\right)}{\zeta(5s+1)}, & m = 2n \\ \frac{q^{-(m-n)(5s+1)}q^{m-2n}\left(1+q^{1-5s}\right)}{\zeta(5s+1)}, & m > 2n \\ \frac{q^{-n(5s+1)}q^{2n-m}\left(1+q^{1-5s}\right)}{\zeta(5s+1)}, & m < 2n \end{cases}$$

Recall that G is embedded in $SO(7)$ and let $\Gamma : \mathrm{G} \to \mathbb{R}_{+}$ be defined by

$$\Gamma(g) = \max_{1\le i,j\le 7} \left|g_{i,j}\right|_{\mathrm{F}}$$

Lemma 6.4. *We have*

1. Γ *is bi*-K-*invariant.*

2. *If* $t_{n,m} = \left(n\alpha^{\vee} + m\beta^{\vee}\right)(\varpi) \in \mathrm{A}$ *then* $\Gamma\left(t_{n,m}\right) = q^{n}$.

3. *For* $g \in \mathrm{G}$*:* $\Gamma(g) = q^{n}$ *if and only if* $g \in \mathrm{K}\, t_{n,m}\, \mathrm{K}$ *for some* $t_{n,m} \in \mathrm{A}$.

Proof. 1. Any element of kg is of the form $\sum_{t=1}^{n} k_{it} g_{tj}$. We choose $L = \sum_{t=1}^{n} k_{it} g_{tj}$ to be the maximal element of kg. Since F is non-Archimedean and $k_{i,t} \in \mathcal{O}$ we have

$$\left|\Gamma(kg)\right|_{\mathrm{F}} \le \max_{t} \left|k_{i,t} g_{t,j}\right|_{\mathrm{F}} \le \max_{t} \left|g_{t,j}\right|_{\mathrm{F}} \le \Gamma(g).$$

On the other hand, since K is a group then $\left|\Gamma(g)\right|_{\mathrm{F}} = \left|\Gamma\left(k^{-1}kg\right)\right|_{\mathrm{F}} \le \Gamma(kg)$ and therefore $\left|\Gamma(g)\right|_{\mathrm{F}} = \left|\Gamma(kg)\right|_{\mathrm{F}}$. Right invariance follows from a symmetrical argument.

2. The element $t_{n,m}$ is of the form

$$\mu = \begin{pmatrix} \varpi^n &&&&&& \\ & \varpi^{m-n} &&&&& \\ && \varpi^{2n-m} &&&& \\ &&& 1 &&& \\ &&&& \varpi^{m-2n} && \\ &&&&& \varpi^{n-m} & \\ &&&&&& \varpi^{-n} \end{pmatrix} .$$

By the assumptions $n \geq m-n \geq 2n-m$ and therefore

$$\Gamma(\mu) = q^n .$$

3. The assertion follows immediately from 1 and 2

□

If we define

$$D_k(t_{n,m}) = \begin{cases} q^{-n}, & n = k \\ 0, & n \neq k \end{cases} ,$$

we can write

$$D_s = \sum_{k=0}^{\infty} D_k q^{-5ks}$$

and thus

$$D_s^{\psi} = \sum_{k=0}^{\infty} D_k^{\psi} q^{-5ks} . \tag{6.1}$$

We can rewrite D_k in terms of Lemma 6.4 as follows:

$$D_k(g) = \begin{cases} q^{-k}, & \Gamma(g) = q^k \\ 0, & otherwise \end{cases} .$$

Now we turn to give a more precise description of the unipotent group U. An element $u \in U$ is of the form

$$u = u(r_1, r_2, r_3, r_4, r_5) = x_{\beta}(r_1)\, x_{\alpha+\beta}(r_2)\, x_{2\alpha+\beta}(r_3)\, x_{3\alpha+\beta}(r_4)\, x_{3\alpha+2\beta}(r_5) .$$

Under the embedding of G_2 in $SO(7)$ the matrix of such u is of the form:

$$u(r_1,r_2,r_3,r_4,r_5)=\begin{pmatrix} 1 & 0 & r_2 & r_3 & -\frac{r_4}{2} & \frac{r_2r_3+r_5}{2} & \frac{-r_3^2+r_2r_4}{2} \\ 0 & 1 & r_1 & r_2 & -\frac{r_3}{2} & \frac{-r_2^2+r_3r_1}{2} & -\frac{r_2r_3+r_5}{2}-\frac{r_2r_3-r_1r_4}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{r_3}{2} & \frac{r_4}{2} \\ 0 & 0 & 0 & 1 & 0 & -r_2 & -r_3 \\ 0 & 0 & 0 & 0 & 1 & -r_1 & -r_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For such u we have $\psi(u)=\psi(r_2+r_3)=\psi(r_2)\psi(r_3)$. We also note that as a unipotent group U is unimodular and the Haar measure du of U is the product measure $du=\prod_{i=1}^{5}dr_i$. Fubini's theorem applies to our integrals since these functions are compactly supported and continuous. In order to avoid complications we shall henceforth assume that the characteristic of the residue field $\mathcal{O}/(\varpi)$ is not 2 or 3 and therefore $|2|_{\mathrm{F}}=|3|_{\mathrm{F}}=1$.

We now turn to compute the Fourier transform $D_k^{\psi}(g)$, which is a function in $C((\mathrm{U},\psi)\setminus \mathrm{G}/\mathrm{K})$. Hence we need to compute its values only on the Levi subgroup $\mathrm{M}\cong \mathrm{GL}(2)$ which in turn has an Iwasawa decomposition $\mathrm{M}=\mathrm{N}'\mathrm{T}'\mathrm{K}'$. Since $\mathrm{M}=\mathrm{GL}_{\alpha}(2)$ and since $\mathrm{K}'\subset \mathrm{K}$ we may assume $g=(t_1,t_2)x_{\alpha}(d)$. More specifically, by left K-invariance we may assume that $(t_1,t_2)=t_{n,m}=(\varpi^n,\varpi^m)$. In the terms of Lemma 6.4 we may write

$$D_k^{\psi}(g)=\int_{\mathrm{U}}D_k(ug)\overline{\psi(u)}du=q^{-k}\int_{u:\Gamma(ug)=q^k}\overline{\psi(u)}du\,.$$

For any $g=t_{n,m}x_{\alpha}(d)$ if $D_k^{\psi}(g)\neq 0$ we must have $k\geq |n|,|m-n|,|2n-m|$. Henceforth we assume that $D_k^{\psi}(g)\neq 0$.

Let

$$U_k(g)=\left\{u\in \mathrm{U}\,\middle|\,\Gamma(ug)\leq q^k\right\}\,.$$

Also denote

$$E_k(g)=\int_{U_k(g)}\overline{\psi(u)}du\,. \tag{6.2}$$

Since the valuation on F is discrete we have

$$D_k^{\psi}(g) = q^{-k}\left(E_k(g) - E_{k-1}(g)\right) . \tag{6.3}$$

Our problem then reduces to calculating E_k which is less complicated. The input g can be either a toral or a non-toral element. We compute $E_k(g)$ for toral elements following these two steps. Similar strategy can be applied to compute E_k for non-toral elements.

1. Let
$$\overline{U_k(g)} = \left\{u(r_1, r_2, r_3, r_4, r_5) \in U_k(g) \middle| \, |r_2|_F, |r_3|_F \le q\right\}$$
and prove that
$$\int_{U_k(g)} \overline{\psi(u)} du = \int_{\overline{U_k(g)}} \overline{\psi(u)} du .$$

2. Denote
$$\begin{aligned}
U_k^{(1)}(g) &= \left\{u(r_1, r_2, r_3, r_4, r_5) \in U_k(g) \middle| \, |r_2|_F, |r_3|_F \le 1\right\}\\
U_k^{(2)}(g) &= \left\{u(r_1, r_2, r_3, r_4, r_5) \in U_k(g) \middle| \, |r_2|_F = q, \ |r_3|_F \le 1\right\}\\
U_k^{(3)}(g) &= \left\{u(r_1, r_2, r_3, r_4, r_5) \in U_k(g) \middle| \, |r_2|_F \le 1, |r_3|_F = q\right\}\\
U_k^{(4)}(g) &= \left\{u(r_1, r_2, r_3, r_4, r_5) \in U_k(g) \middle| \, |r_2|_F, |r_3|_F = q\right\}
\end{aligned}$$
and calculate
$$E_k^{(i)}(g) = \int_{U_k^{(i)}(g)} \overline{\psi(u)} du .$$
We get
$$E_k(g) = \sum_{i=1}^{4} E_k^{(i)}(g) .$$

We apply this method for toral elements. Assume $g = (t_1, t_2) = \alpha^\vee(t_1)\beta^\vee(t_2)$ and denote $|t_1|_F = q^{-n}$ and $|t_2|_F = q^{-m}$. By Proposition 6.1 (with $d = 0$) we may assume that $\varpi^n, \varpi^{m-n} \in \mathcal{O}$ but this simply yields $m \ge n \ge 0$. The following holds:

$$m - n \ge 0, \quad n \ge 2n - m, \quad m - n \ge m - 2n .$$

We also assume that $k \geq n, m-n$ since for other k we know $E_k(t_{n,m})$ to vanish. The condition $\Gamma(ug) \leq q^k$ can be reduced to the following inequalities

$$|r_2|_F, |r_3|_F, |r_4|_F, \left|r_2r_4 - r_3^2\right|_F, |r_1r_4 - r_5 - 2r_2r_3|_F \leq q^{k-n}$$
$$|r_2|_F, |r_3|_F, |r_1|_F, |r_2r_3 + r_5|_F, \left|r_1r_3 - r_2^2\right|_F \leq q^{k+n-m} .$$

By Fubini's theorem we may write

$$E_k(g) = \int_F \overline{\psi(r_2)} \int_F \overline{\psi(r_3)} V(r_2, r_3)\, dr_2\, dr_3 .$$

Lemma 6.5. *For any $s, t \in O^\times$ the following equalities hold*

$$V(sr_2, tr_3) = V(r_2, r_3) .$$

Proof. We have

$$E_k(t_{n,m}) = \sum_{i,j=-\infty}^{\infty} \int_{\varpi^i O^\times} \overline{\psi(r_2)} \int_{\varpi^j O^\times} \overline{\psi(r_3)} V(r_2, r_3)\, dr_2\, dr_3 .$$

The function $V(x,y)$ is the volume of the set

$$U_{x,y} = \{u(r_1, r_2, r_3, r_4, r_5) \in U_k(g) |\, r_2 = x,\ r_3 = y\} .$$

For any $u(r_1, x, y, r_4, r_5) \in U_{x,y}$ we have

$$u\left(\frac{s^2}{t} r_1, sx, ty, \frac{t^2}{s} r_4, tsr_5\right) \in U_{sx,ty} .$$

We recall that for a Haar measure on F we have $d(sx) = |s|_F\, dx$. From Fubini's theorem we get

$$V(x,y) = \int_{U_{x,y}} d'r_1 dr_4' dr_5' = \int_{U_{sx,ty}} dr_1 dr_4 dr_5 = V(sx, ty) .$$

□

Let

$$\overline{U_k(g)} = \{u(r_1, r_2, r_3, r_4, r_5) \in U_k(g) |\, |r_2|_F, |r_3|_F \leq q\} .$$

Lemma 6.6.

$$\int_{U_k(g)} \overline{\psi(u)} du = \int_{\overline{U_k(g)}} \overline{\psi(u)} du \,.$$

Proof. We have

$$E_k(g) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \int_{\varpi^i O} \overline{\psi(r_2)} \int_{\varpi^j O} \overline{\psi(r_3)} V(r_2, r_3)\, dr_2\, dr_3$$

By the previous lemma we know that V depends only on the valuations of r_2 and r_3. Denote by $V(i, j)$ the value $V(r_2, r_3)$ attains for $val(r_2) = i$ and $val(r_3) = j$. Using this notation we have

$$E_k(g) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} V(i, j) \int_{\varpi^i O} \overline{\psi(r_2)} dr_2 \int_{\varpi^j O} \overline{\psi(r_3)} dr_3 \,.$$

Since ψ is of conductor O, the integral $\int_{\varpi^a O^\times} \overline{\psi(r)} dr$ vanishes for any $a < -1$ and thus our assertion is proved. □

We summarize the results of case 1 in the following proposition

Proposition 6.3. *If $m - n = n$ we have*

$$E_k(t_{n,m}) = \begin{cases} 1, & k = n = m - n \\ 2q^2, & k = n + 1 = m - n + 1 \\ 0, & k > n + 1 \ \text{or} \ k < n \end{cases}, \tag{6.4}$$

if $m - n > n$ we have

$$E_k(t_{n,m}) = \begin{cases} q^{m-2n}, & k = m - n \\ q^2 q^{m-2n}, & k = m - n + 1 \\ 0, & k > m - n + 1 \ \text{or} \ k < m - n \end{cases} \tag{6.5}$$

and if $m - n < n$ we have

$$E_k(t_{n,m}) = \begin{cases} q^{2n-m}, & k = n \\ q^2 q^{2n-m}, & k = n + 1 \\ 0, & k > n + 1 \ \text{or} \ k < n \end{cases}. \tag{6.6}$$

Proof. We split this calculation into the following cases:

Case 1: Assume that $m = 2n$, i.e. $|\alpha(t_1, t_2)|_F = 1$.

Case 1.1: Assume $k = n$. In this case the condition $\Gamma(ug) \leq q^n$ is simply

$$|r_1|_F, |r_2|_F, |r_3|_F, |r_4|_F, |r_5|_F \leq 1,$$

from which follows

$$E_n(t_{n,2n}) = 1.$$

Case 1.2: Assume $k = n+1 = m-n+1$. In this case the condition $\Gamma(ug) \leq q^k$ is simply

$$\begin{aligned} & |r_1|_F, |r_2|_F, |r_3|_F, |r_4|_F \leq q \\ & \left|r_2 r_4 - r_3^2\right|_F, |r_1 r_4 - r_5 - 2r_2 r_3|_F, |r_2 r_3 + r_5|_F, \left|r_1 r_3 - r_2^2\right|_F \leq q. \end{aligned}$$

Analysing these conditions we see that when $|r_2|_F = q$ and $|r_3|_F \leq 1$ then $|r_1 r_3|_F \leq q$ and therefore $\left|r_1 r_3 - r_2^2\right|_F = q^2$ which contradicts the above inequalities. By a similar argument we can't have $|r_3|_F = q$ and $|r_2|_F \leq 1$.

When $|r_2|_F, |r_3|_F \leq 1$ then all the above inequalities reduce to

$$\begin{aligned} & |r_1|_F, |r_2|_F, |r_3|_F, |r_4|_F, |r_5|_F \leq q \\ & |r_1 r_4 - r_5|_F \leq q. \end{aligned}$$

The last one says that $|r_1 r_4|_F \leq q$ and that $r_5 = r_1 r_4 + x$ where $|x|_F \leq q$.

Consider, on the other hand, the case when $|r_2|_F, |r_3|_F \leq q$. Then the above inequalities reduce to

$$\begin{aligned} r_4 &= \frac{r_3^2 + x}{r_2}, \quad |x|_F \leq q \\ r_1 &= \frac{r_2^2 + y}{r_3}, \quad |y|_F \leq q \\ r_5 &= z - r_2 r_3, \quad |z|_F \leq q. \end{aligned}$$

A short calculation shows that $|r_1r_4 - r_5 - 2r_2r_3|_F \leq q$ follows from this. All of the above yields

$$E_{n+1}(t_{n,2n}) = \int_O dr_2 \int_O dr_3 \int_{\varpi^{-1}O} dr_5 \left(\int_O dr_1 \int_{\varpi^{-1}O} dr_4 + \int_{\varpi^{-1}O^\times} dr_1 \int_O dr_4 \right) + \\ + \int_{\varpi^{-1}O^\times} \overline{\psi(r_2)} dr_2 \int_{\varpi^{-1}O^\times} \overline{\psi(r_3)} dr_3 \int_{\varpi^{-1}O} q^{-1} dx \int_{\varpi^{-1}O} q^{-1} dy \int_{\varpi^{-1}O} dz \,.$$

We can now compute all of the above integrals simultaneously and we get

$$E_{n+1}(t_{n,2n}) = q\left(q + q\left(1 - q^{-1}\right)\right) + q = 2q^2 \,.$$

Case 1.3: Assume that $k > n+1 = m-n+1$, in which case the condition $\Gamma(ug) \leq q^k$ is simply

$$|r_2|_F \,, |r_3|_F \leq q$$
$$|r_1|_F \,, |r_4|_F \,, |r_5|_F \,, |r_2r_4|_F \,, |r_1r_3|_F \,, |r_1r_4|_F \leq q^{k-n}$$

and then

$$E_k(t_{n,2n}) = \int_O dr_2 \int_O dr_3 \int_{\varpi^{n-k}O} dr_5 \cdot \\ \cdot \left(\int_O dr_1 \int_{\varpi^{n-k}O} dr_4 + \sum_{j=1}^{k-n} \int_{\varpi^{-j}O^\times} dr_1 \int_{\varpi^{n-k+j}O} dr_4 \right) + \\ + \int_{\varpi^{-1}O^\times} \overline{\psi(r_2)} dr_2 \int_O dr_3 \int_{\varpi^{n-k}O} dr_5 \cdot \\ \cdot \left(\int_O dr_1 \int_{\varpi^{n-k+1}O} dr_4 + \sum_{j=1}^{k-n} \int_{\varpi^{-j}O^\times} dr_1 \int_{\varpi^{n-k+j}O} dr_4 \right) + \\ + \int_O dr_2 \int_{\varpi^{-1}O^\times} \overline{\psi(r_3)} dr_3 \int_{\varpi^{n-k}O} dr_5 \cdot \\ \cdot \left(\int_O dr_1 \int_{\varpi^{n-k}O} dr_4 + \sum_{j=1}^{k-n-1} \int_{\varpi^{-j}O^\times} dr_1 \int_{\varpi^{n-k+j}O} dr_4 \right) + \\ + \int_{\varpi^{-1}O^\times} \overline{\psi(r_2)} dr_2 \int_{\varpi^{-1}O^\times} \overline{\psi(r_3)} dr_3 \int_{\varpi^{n-k}O} dr_5 \cdot \\ \cdot \left(\int_O dr_1 \int_{\varpi^{n-k+1}O} dr_4 + \sum_{j=1}^{k-n-1} \int_{\varpi^{-j}O^\times} dr_1 \int_{\varpi^{n-k+j}O} dr_4 \right) = 0 \,.$$

Case 2: Assume $n < m - n$, i.e. $|\alpha(t_1, t_2)|_F < 1$.

Case 2.1: Assume $k = m - n$. In this case the condition $\Gamma(ug) \le q^n$ is simply

$$|r_4|_F \le q^{k-n}$$
$$|r_1|_F, |r_2|_F, |r_3|_F, |r_5|_F \le 1$$

and therefore

$$E_{m-n}(t_{n,m}) = \int_O dr_1 \int_O dr_2 \int_O dr_3 \int_O dr_5 \int_{\varpi^{n-k}O} dr_4 = q^{k-n} = q^{m-2n}.$$

Case 2.2: Assume $k = m - n + 1$, in which case the condition $\Gamma(ug) \le q^k$ is simply

$$q = q^{k-m+n} < q^{k-n}$$
$$|r_4|_F, \left|r_2r_4 - r_3^2\right|_F, |r_1r_4 - r_5 - 2r_2r_3|_F \le q^{k-n}$$
$$|r_1|_F, |r_2|_F, |r_3|_F, |r_2r_3 + r_5|_F, \left|r_1r_3 - r_2^2\right|_F \le q.$$

We can reduce this to the following cases:

$$|r_2|_F, |r_3|_F \le 1 \Rightarrow |r_1|_F, |r_5|_F \le q, \quad |r_4|_F, |r_1r_4|_F \le q^{k-n}$$
$$|r_2|_F \le 1, |r_3|_F = q \Rightarrow |r_1|_F \le 1, \quad |r_5|_F \le q, \quad |r_4|_F \le q^{k-n}$$
$$|r_2|_F, |r_3|_F = q \Rightarrow |r_4|_F \le q^{k-n-1}, \quad r_1 = \frac{y + r_2^2}{r_3}, r_5 = y - r_2r_3, \quad |x|_F, |y|_F \le q.$$

Here we have omitted the case where $|r_2|_F = q$ and $r_3 \in O$ since $\left|r_2^2\right|_F = q^2, |r_1r_3|_F \le q$ and $\left|r_1r_3 - r_2^2\right|_F = q^2$ which contradicts the above inequalities, making this set empty. This yields the following result:

$$\begin{aligned}
E_{m-n+1}(t_{n,m}) = &\int_O dr_2 \int_O dr_3 \int_{\varpi^{-1}O} dr_5 \cdot \\
&\cdot \left(\int_O dr_1 \int_{\varpi^{n-k}O} dr_4 + \int_{\varpi^{-1}O^\times} dr_1 \int_{\varpi^{1+n-k}O} dr_4\right) + \\
&+ \int_O dr_2 \int_{\varpi^{-1}O^\times} \overline{\psi(r_3)} dr_3 \int_{\varpi^{-1}O} dr_5 \int_O dr_1 \int_{\varpi^{n-k}O} dr_4 \\
&+ \int_{\varpi^{-1}O^\times} \overline{\psi(r_2)} dr_2 \int_{\varpi^{-1}O^\times} \overline{\psi(r_3)} dr_3 \int_{\varpi^{1+n-k}O} dr_4 \int_{\varpi^{-1}O} q^{-1} dx \int_{\varpi^{-1}O} dy \\
=& q\left(q^{k-n} + q\left(1 - q^{-1}\right) q^{k-n-1}\right) - qq^{k-n} + q^{k-n-1}q = q^{m-2n+2}.
\end{aligned}$$

Case 2.3: Assume $k > m - n + 1 > n + 1$, in which case the condition $\Gamma(ug) \leq q^k$ is simply

$$|r_2|_F, |r_3|_F \leq q$$
$$|r_4|_F, |r_2 r_4|_F, |r_1 r_4|_F \leq q^{k-n}$$
$$|r_1|_F, |r_5|_F, |r_1 r_3|_F \leq q^{k+n-m}$$

and so

$$\begin{aligned}
E_k(t_{n,m}) = &\int_{\mathcal{O}} dr_2 \int_{\mathcal{O}} dr_3 \int_{\varpi^{m-k-n}\mathcal{O}} dr_5 \cdot \\
&\cdot \left(\int_{\mathcal{O}} dr_1 \int_{\varpi^{n-k}\mathcal{O}} dr_4 + \sum_{j=1}^{k+n-m} \int_{\varpi^{-j}\mathcal{O}^\times} dr_1 \int_{\varpi^{j+n-k}\mathcal{O}} dr_4 \right) + \\
+ &\int_{\varpi^{-1}\mathcal{O}^\times} \overline{\psi(r_2)} dr_2 \int_{\mathcal{O}} dr_3 \int_{\varpi^{m-k-n}\mathcal{O}} dr_5 \cdot \\
&\cdot \left(\int_{\mathcal{O}} dr_1 \int_{\varpi^{n-k+1}\mathcal{O}} dr_4 + \sum_{j=1}^{k+n-m} \int_{\varpi^{-j}\mathcal{O}^\times} dr_1 \int_{\varpi^{j+n-k}\mathcal{O}} dr_4 \right) + \\
+ &\int_{\mathcal{O}} dr_2 \int_{\varpi^{-1}\mathcal{O}^\times} \overline{\psi(r_3)} dr_3 \int_{\varpi^{m-k-n}\mathcal{O}} dr_5 \cdot \\
&\cdot \left(\int_{\mathcal{O}} dr_1 \int_{\varpi^{n-k}\mathcal{O}} dr_4 + \sum_{j=1}^{k+n-m-1} \int_{\varpi^{-j}\mathcal{O}^\times} dr_1 \int_{\varpi^{j+n-k}\mathcal{O}} dr_4 \right) + \\
+ &\int_{\varpi^{-1}\mathcal{O}^\times} \overline{\psi(r_2)} dr_2 \int_{\varpi^{-1}\mathcal{O}^\times} \overline{\psi(r_3)} dr_3 \int_{\varpi^{m-k-n}\mathcal{O}} dr_5 \cdot \\
&\cdot \left(\int_{\mathcal{O}} dr_1 \int_{\varpi^{n-k+1}\mathcal{O}} dr_4 + \sum_{j=1}^{k+n-m-1} \int_{\varpi^{-j}\mathcal{O}^\times} dr_1 \int_{\varpi^{j+n-k}\mathcal{O}} dr_4 \right) = 0 .
\end{aligned}$$

Case 3: Assume $m - n < n$, i.e. $|\alpha(t_1, t_2)|_F > 1$.

Case 3.1: Assume $k = n$, in which case the condition $\Gamma(ug) \leq q^n$ is simply

$$|r_2|_F, |r_3|_F, |r_4|_F, \left|r_2 r_4 - r_3^2\right|_F, |r_1 r_4 - r_5 - 2 r_2 r_3|_F \leq 1$$
$$|r_1|_F, |r_2 r_3 + r_5|_F, \left|r_1 r_3 - r_2^2\right|_F \leq q^{2n-m} .$$

Since $r_2, r_3, r_4 \in \mathcal{O}$ and $1 \leq q^{2n-m}$, $\left|r_2 r_4 - r_3^2\right|_F \leq 1$ and $\left|r_1 r_3 - r_2^2\right|_F \leq q^{2n-m}$ are trivial and $|r_2 r_3 + r_5|_F \leq q^{2n-m}$ reduce to $|r_5|_F \leq q^{2n-m}$.

The last inequality we need to understand is $|r_1r_4 - r_5 - 2r_2r_3|_F \leq 1$. Since $2r_2r_3 \in \mathcal{O}$, the inequality is equivalent to $|r_1r_4 - r_5|_F \leq 1$. If $|r_1|_F \leq 1$ then obviously $|r_1r_4|_F \leq 1$, and then $|r_5|_F \leq 1$; otherwise $|r_1|_= q^j, j > 0$ and then we have two options, if $|r_1|_F \leq q^{-j}$ then $|r_5|_F \leq 1$ and if not $|r_1|_F > q^{-j}$ in which case $r_5 = r_1r_4 + x$ with $|x|_F \leq 1$. Recall now that $dr_5 = dx$ since it is a Haar measure. We can now compute

$$E_n\left(t_{n,m}\right) = \int_{\mathcal{O}} dr_2 \int_{\mathcal{O}} dr_3 \int_{\mathcal{O}} dr_5 \int_{\mathcal{O}} dr_4 \int_{\varpi^{m-2n}\mathcal{O}} dr_1 = q^{2n-m} .$$

Case 3.2: Assume $k = n+1$. In this case the condition $\Gamma(ug) \leq q^k$ is simply

$$\begin{aligned} & q = q^{k-n} < q^{k-m+n} \\ & |r_2|_F, |r_3|_F, |r_4|_F, \left|r_2r_4 - r_3^2\right|_F, |r_1r_4 - r_5 - 2r_2r_3|_F \leq q \\ & |r_1|_F, |r_5|_F, |r_1r_3|_F \leq q^{k+n-m} . \end{aligned}$$

We can reduce this to the following cases:

$$\begin{aligned} |r_2|_F, |r_3|_F \leq 1 &\Rightarrow |r_1|_F, |r_1r_4|_F \leq q^{k-m+n},\ |r_4|_F \leq q,\ r_5 = x + r_1r_4,\ |x|_F \leq q \\ |r_2|_F = q, |r_3|_F \leq 1 &\Rightarrow |r_1|_F \leq q^{k-m+n},\ |r_4|_F \leq 1,\ r_5 = x + r_1r_4,\ |x|_F \leq q \\ |r_2|_F, |r_3|_F = q &\Rightarrow |r_1|_F \leq q^{k-m+n-1},\ r_4 = \frac{r_3^2 + y}{r_2},\ r_5 = x + r_1r_4 - 2r_2r_3,\ |x|_F, |y|_F \leq q . \end{aligned}$$

Here we have omitted the case where $|r_3|_F = q$ and $r_2 \in \mathcal{O}$ since $\left|r_3^2\right|_F = q^2, |r_2r_4|_F \leq q$ and $\left|r_2r_4 - r_3^2\right|_F = q^2$, which contradicts the above inequalities making this set empty. This yields the following result:

$$\begin{aligned} E_{n+1}\left(t_{n,m}\right) = & \int_{\mathcal{O}} dr_2 \int_{\mathcal{O}} dr_3 \int_{\varpi^{-1}\mathcal{O}} dx \left(\int_{\mathcal{O}} dr_4 \int_{\varpi^{m-n-k}\mathcal{O}} dr_1 + \int_{\varpi^{-1}\mathcal{O}^\times} dr_4 \int_{\varpi^{m-n-k+1}\mathcal{O}} dr_1 \right) + \\ & + \int_{\varpi^{-1}\mathcal{O}^\times} \overline{\psi\left(r_2\right)} dr_2 \int_{\mathcal{O}} dr_3 \int_{\varpi^{-1}\mathcal{O}} dx \int_{\mathcal{O}} dr_4 \int_{\varpi^{m-n-k}\mathcal{O}} dr_1 + \\ & + \int_{\varpi^{-1}\mathcal{O}^\times} \overline{\psi\left(r_2\right)} dr_2 \int_{\varpi^{-1}\mathcal{O}^\times} \overline{\psi\left(r_3\right)} dr_3 \int_{\varpi^{-1}\mathcal{O}} dx \int_{\varpi^{-1}\mathcal{O}^\times} q^{-1} dy \int_{\varpi^{m-n-k+1}\mathcal{O}} dr_1 = \\ & = q\left(q^{k-m+n} + q\left(1 - q^{-1}\right) q^{k-m+n-1}\right) - qq^{k-m+n} + qq^{k-m+n-1} = q^{2n-m+2} . \end{aligned}$$

Case 3.3: Assume $k > n > m - n$, in which case the condition $\Gamma(ug) \leq q^k$ is simply

$$\begin{aligned}
&|r_2|_{\mathrm{F}}, |r_3|_{\mathrm{F}} \leq q \\
&|r_4|_{\mathrm{F}}, |r_2 r_4|_{\mathrm{F}} \leq q^{k-n} \\
&|r_1|_{\mathrm{F}}, |r_1 r_4|_{\mathrm{F}} \leq q^{k+n-m} \\
&r_5 = x + r_1 r_4, \quad |x|_{\mathrm{F}} \leq q^{k-n}
\end{aligned}$$

and therefore

$$\begin{aligned}
E_k(t_{n,m}) = &\int_{\mathcal{O}} dr_2 \int_{\mathcal{O}} dr_3 \int_{\varpi^{n-k}\mathcal{O}} dr_5 \cdot \\
&\cdot \left(\int_{\mathcal{O}} dr_4 \int_{\varpi^{m-n-k}\mathcal{O}} dr_1 + \sum_{j=1}^{k-n} \int_{\varpi^{-j}\mathcal{O}^\times} dr_4 \int_{\varpi^{m-n-k+j}\mathcal{O}} dr_1 \right) + \\
&+ \int_{\varpi^{-1}\mathcal{O}^\times} \overline{\psi(r_2)} dr_2 \int_{\mathcal{O}} dr_3 \int_{\varpi^{n-k}\mathcal{O}} dr_5 \cdot \\
&\cdot \left(\int_{\mathcal{O}} dr_4 \int_{\varpi^{m-n-k}\mathcal{O}} dr_1 + \sum_{j=1}^{k-n-1} \int_{\varpi^{-j}\mathcal{O}^\times} dr_4 \int_{\varpi^{m-n-k+j}\mathcal{O}} dr_1 \right) + \\
&+ \int_{\mathcal{O}} dr_2 \int_{\varpi^{-1}\mathcal{O}^\times} \overline{\psi(r_3)} dr_3 \int_{\varpi^{n-k}\mathcal{O}} dr_5 \cdot \\
&\cdot \left(\int_{\mathcal{O}} dr_4 \int_{\varpi^{m-n-k+1}\mathcal{O}} dr_1 + \sum_{j=1}^{k-n} \int_{\varpi^{-j}\mathcal{O}^\times} dr_4 \int_{\varpi^{m-n-k+j}\mathcal{O}} dr_1 \right) + \\
&+ \int_{\varpi^{-1}\mathcal{O}^\times} \overline{\psi(r_2)} dr_2 \int_{\varpi^{-1}\mathcal{O}^\times} \overline{\psi(r_3)} dr_3 \int_{\varpi^{n-k}\mathcal{O}} dr_5 \cdot \\
&\cdot \left(\int_{\mathcal{O}} dr_4 \int_{\varpi^{m-n-k+1}\mathcal{O}} dr_1 + \sum_{j=1}^{k-n-1} \int_{\varpi^{-j}\mathcal{O}^\times} dr_4 \int_{\varpi^{m-n-k+j}\mathcal{O}} dr_1 \right) = 0\,.
\end{aligned}$$

□

Chapter 7

Computing $F_S * P_S$

Let $\mathrm{G} = \mathrm{G}_2(\mathrm{F})$, $\mathrm{K} = \mathrm{G}(\mathcal{O})$ be its usual maximal compact subgroup, B be the Borel subgroup and P be the Heisenberg parabolic subgroup. Let $\mathrm{H} = Spin_8$ with $\mathrm{P_H}$ the parabolic subgroup described in chapter 1.

Also, let ${}^L\mathrm{G} = \mathrm{G}_2(\mathbb{C})$ be the Langlands dual group of G_2. Denote by ω_1,ω_2 the first and second fundamental weights of ${}^L\mathrm{G}$. We denote the finite-dimensional irreducible representation of ${}^L\mathrm{G}$ with $n\omega_1 + m\omega_2$ as its highest weight by $V_{[n,m]}$. Let $A_{[n,m]}$ be the corresponding function in $\mathcal{H}$ attached to $V_{[n,m]}$ by Satake correspondence.

Let $f_s \in \mathrm{Ind}_{\mathrm{P_H}}^{\mathrm{H}}(\delta_{\mathrm{P_H}}^s)$ be a spherical vector normalized by $f_s(e) = 1$.

7.1 Unramified Computation of F_s

We now consider the function F_s described in Theorem 4.1 as

$$F_s(g) = \int_{\mathrm{F}} f_s\left(w_2 w_3 x_{-\alpha_1}(1) x_{\alpha+\beta}(r)\, g\right) \psi(r)\, dr\ .$$

Since it is left-(U,ψ)-invariant and right-K-invariant, then by Levi decomposition we can think of F_s as a function on the Levi subgroup of P which is isomorphic to $\mathrm{GL}(2)$. By Iwasawa decomposition, we can write each element in $\mathrm{GL}(2)$ as

$$g = (t_1, t_2)\, x_\alpha(s)\, k : \quad t_1, t_2, s \in F, k \in \mathrm{K}(\mathrm{M})\ .$$

Since $K(M) \subset K$ and F_s is right K-invariant, we may assume further that $k = 1$. In this section we calculate $F_s(g)$, the results are summarized in the following proposition:

Proposition 7.1. *We have*

$$F_s((t_1,t_2)x_\alpha(d)) = \begin{cases} \frac{1}{\zeta(5s)}, & t_1 = t_2 = 1, \quad d = 0 \\ |t_1|_F^{5s}\left|t_1^{-1}t_2\right|_F \frac{\zeta(5s-1)}{\zeta(5s)}\left(1 - \left|\frac{t_2}{t_1}\right|_F^{5s-1} q^{-(5s-1)}\right), & \left|t_1^2t_2^{-1}\right|_F \le 1, \quad d = 0 \\ |t_1|_F\left|t_1^{-1}t_2\right|_F^{5s} \frac{\zeta(5s-1)}{\zeta(5s)}\left(1 - |t_1|_F^{5s-1} q^{-(5s-1)}\right), & \left|t_1^2t_2^{-1}\right|_F > 1, \quad d = 0 \\ |t_1|_F^{5s}\left|t_1^{-1}t_2\right|_F \frac{\zeta(5s-1)}{\zeta(5s)}\left(1 - \left|\frac{t_2}{t_1}\right|_F^{5s-1} q^{-(5s-1)}\right), & |p|_F \le 1, \quad d \notin \mathcal{O} \\ \left|t_1p^{-1}\right|_F^{5s}\left|t_1^{-1}t_2p\right|_F \frac{\zeta(5s-1)}{\zeta(5s)}\left(1 - \left|\frac{pt_2}{t_1}\right|_F^{5s-1} q^{-(5s-1)}\right), & \left|\frac{t_2}{t_1}p\right|_F \le 1, \quad d \notin \mathcal{O} \\ 0, & \textit{otherwise} \end{cases},$$

where $p := \frac{t_1^2}{t_2}d^2 + d$.

The Iwasawa decomposition of $x_{-\alpha}(r)$ can be computed by transferring the calculation to the $SL(2)$ group generated by α:

Lemma 7.1. *For any* $\alpha \in \Phi^+$ *and* $|r|_F > 1$ *we have*

$$x_{-\alpha}(r) = x_\alpha\left(r^{-1}\right)\alpha^\vee\left(r^{-1}\right)k, \quad k \in K$$
$$x_\alpha(r) = \alpha^\vee(r)\,x_{-\alpha}(r)\,k, \quad k \in K\ .$$

Let us proof Proposition 7.1

Proof. **Case I:** Assume that $g = 1$

$$F_s(g) = \int_F f_s\left(w_2w_3x_{-\alpha_1}(1)x_{\alpha+\beta}(r)\,g\right)\psi(r)\,dr =$$
$$= \int_F f_s\left(w_2w_3x_{-(1000)}(1)x_{(1100)}(r)\,x_{(0101)}(r)\,x_{(0110)}(r)\right)\psi(r)\,dr =$$
$$= \int_F f_s\left(w_2w_3x_{(0100)}(-r)\,x_{(1100)}(r)\,x_{(0101)}(r)\,x_{(0110)}(r)\,x_{-(1000)}(1)\right)\psi(r)\,dr =$$
$$= \int_F f_s\left(w_2w_3x_{(0100)}(-r)\,x_{(1100)}(r)\,x_{(0101)}(r)\,x_{(0110)}(r)\right)\psi(r)\,dr =$$
$$= \int_F f_s\left(x_{(0010)}(-r)\,x_{(1110)}(r)\,x_{(0111)}(r)\,w_2x_{(0100)}(r)\,w_3\right)\psi(r)\,dr\ .$$

By right-K-invariance:

$$
\begin{aligned}
F_s(g) &= \int_{\mathrm{F}} f_s\left(x_{-(0100)}(r)\right)\psi(r)\,dr = \\
&= \int_{O} f_s\left(x_{-(0100)}(r)\right)\psi(r)\,dr + \int_{F\backslash O} f_s\left(x_{-(0100)}(r)\right)\psi(r)\,dr = \\
&= \int_{O} f_s(1)\,\psi(r)\,dr + \int_{F\backslash O} f_s\left(x_{(0100)}\left(r^{-1}\right)\alpha_2^{\vee}\left(r^{-1}\right)k(r)\right)\psi(r)\,dr = \\
&= \int_{O} 1dr + \int_{F\backslash O} |r|_F^{-5s}\,\psi(r)\,dr = 1 - q^{-5s}\,.
\end{aligned}
$$

The last equality is due to the fact that ψ is of conductor O.

<u>Case II:</u> Assume $g = (t_1, t_2)$. By Proposition 6.1 we may assume that $m \geq n \geq 0$. We can then deduce also that $m - n \geq 0,\ n \geq 2n - m$. We then compute $F_s(t_{n,m})$

$$
\begin{aligned}
F_s(g) &= \int_{\mathrm{F}} f_s\left(w_2 w_3 x_{-\alpha_1}(1) x_{\alpha+\beta}(r)\, g\right)\psi(r)\,dr = \\
&= \int_{\mathrm{F}} f_s\left(w_2 w_3\; g\; x_{-\alpha_1}\left(\frac{1}{\alpha_1\left(g^{-1}\right)}\right) x_{\alpha+\beta}\left(\left(\alpha+\beta\right)\left(g^{-1}\right) r\right)\right)\psi(r)\,dr = \\
&= \int_{\mathrm{F}} f_s\left(w_2 w_3\; g\; x_{-\alpha_1}\left(t_1^2 t_2^{-1}\right) x_{\alpha+\beta}\left(t_1 t_2^{-1} r\right)\right)\psi(r)\,dr = \left\{y = t_1 t_2^{-1} r\right\} = \\
&= \left|t_1^{-1} t_2\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(g^{w_3 w_2} w_2 w_3 x_{-\alpha_1}\left(t_1^2 t_2^{-1}\right) x_{\alpha+\beta}(y)\right)\psi\left(t_1^{-1} t_2 y\right)dy = \\
&= \delta_{\mathrm{P}}\left(g^{w_3 w_2}\right)^s \left|t_1^{-1} t_2\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(w_2 w_3 x_{-\alpha_1}\left(t_1^2 t_2^{-1}\right) x_{\alpha+\beta}(y)\right)\psi\left(t_1^{-1} t_2 y\right)dy\,.
\end{aligned}
$$

As in the prior case we can get rid of $x_{(1100)}(r)$ and $x_{(0101)}(r)$ by moving them to the left and using the left U-invariance of f_s we get

$$
F_s(g) = |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(w_2 w_3 x_{-\alpha_1}\left(t_1^2 t_2^{-1}\right) x_{(0110)}(y)\right)\psi\left(t_1^{-1} t_2 y\right)dy\,.
$$

Here we must consider two options:

Option I: Assume that $\left|t_1^2 t_2^{-1}\right|_{\mathrm{F}} \leq 1$, so $x_{-\alpha_1}\left(t_1^2 t_2^{-1}\right) \in \mathrm{K}$ and we have

$$\begin{aligned}
F_s(g) &= |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(w_2 w_3 x_{(0100)}\left(-t_1^2 t_2^{-1} y\right) x_{(0110)}(y)\right) \psi\left(t_1^{-1} t_2 y\right) dy = \\
&= |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(x_{(0010)}\left(t_1^2 t_2^{-1} y\right) w_2 x_{(0100)}(y) w_3\right) \psi\left(t_1^{-1} t_2 y\right) dy = \\
&= |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(w_2 x_{(0100)}(y)\right) \psi\left(t_1^{-1} t_2 y\right) dy .
\end{aligned}$$

By separation into the integral and non-integral parts

$$\begin{aligned}
F_s(g) &= |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \int_{\mathcal{O}} f_s\left(w_2 x_{(0100)}(y)\right) \psi\left(t_1^{-1} t_2 y\right) dy + \\
&+ |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \int_{\mathrm{F}\setminus\mathcal{O}} f_s\left(w_2 x_{(0100)}(y)\right) \psi\left(t_1^{-1} t_2 y\right) dy = \\
&= |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \int_{\mathcal{O}} \psi\left(t_1^{-1} t_2 y\right) dy + \\
&+ |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \int_{\mathrm{F}\setminus\mathcal{O}} f_s\left(\alpha_2^\vee\left(y^{-1}\right) x_{(0100)}(y)\right) \psi\left(t_1^{-1} t_2 y\right) dy = \\
&= |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \left(\int_{\mathcal{O}} \psi\left(t_1^{-1} t_2 y\right) dy + \int_{\mathrm{F}\setminus\mathcal{O}} f_s\left(\alpha_2^\vee\left(y^{-1}\right)\right) \psi\left(t_1^{-1} t_2 y\right) dy\right) .
\end{aligned}$$

Since ψ is of conductor $\mathcal{O}$ we get

$$\begin{aligned}
F_s(g) &= |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \left(1 + \sum_{k=1}^{\infty} \int_{\varpi^{-k}\mathcal{O}^\times} |r|_{\mathrm{F}}^{-5s} \psi\left(t_1^{-1} t_2 y\right) dy\right) = \\
&= |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \left(1 + \sum_{k=1}^{\infty} q^{-5ks} \int_{\varpi^{-k}\mathcal{O}^\times} \psi(y) \frac{dy}{\left|t_1^{-1} t_2\right|_{\mathrm{F}}}\right) = \\
&= |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \left(1 + \sum_{k=1}^{\infty} q^{-5ks} q^{k}\left(1 - q^{-1}\right) - q^{-5s(m+1)} q^{m}\right) = \\
&= |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \left(1 + \sum_{k=1}^{\infty} q^{-k(5s-1)} - q^{-m(5s-1)} q^{-5s}\right) = \\
&= |t_1|_{\mathrm{F}}^{5s} \left|t_1^{-1} t_2\right|_{\mathrm{F}} \frac{\zeta(5s-1)}{\zeta(5s)} \left(1 - \left|\frac{t_2}{t_1}\right|_{\mathrm{F}}^{5s-1} q^{-(5s-1)}\right) .
\end{aligned}$$

Option II: Assume that $\left|t_1^2t_2^{-1}\right|_F > 1$, in which case

$$\begin{aligned}
&f_s\left(w_2w_3x_{-\alpha_1}\left(t_1^2t_2^{-1}\right)x_{(0110)}(y)\right) = f_s\left(w_2w_3x_{(0110)}(y)\,x_{-\alpha_1}\left(t_1^2t_2^{-1}\right)\right) = \\
&f_s\left(w_2w_3x_{(0110)}(y)\,x_{\alpha_1}\left(t_1^{-2}t_2\right)\alpha_1^\vee\left(t_1^{-2}t_2\right)k\right) = \\
&f_s\left(w_2w_3x_{(0110)}(y)\,x_{\alpha_1}\left(t_1^{-2}t_2\right)\alpha_1^\vee\left(t_1^{-2}t_2\right)\right) = \\
&f_s\left(w_2w_3x_{(1110)}\left(-t_1^{-2}t_2y\right)x_{\alpha_1}\left(t_1^{-2}t_2\right)x_{(0110)}(y)\,\alpha_1^\vee\left(t_1^{-2}t_2\right)\right) = \\
&f_s\left(x_{\alpha_1}\left(-t_1^{-2}t_2y\right)x_{(1100)}\left(t_1^{-2}t_2\right)w_2w_3x_{(0110)}(y)\,\alpha_1^\vee\left(t_1^{-2}t_2\right)\right) = \\
&f_s\left(w_2w_3x_{(0110)}(y)\,\alpha_1^\vee\left(t_1^{-2}t_2\right)\right) = f_s\left(w_2w_3\alpha_1^\vee\left(t_1^{-2}t_2\right)x_{(0110)}\left(t_1^{-2}t_2y\right)\right) = \\
&f_s\left((1100)^\vee\left(t_1^{-2}t_2\right)w_2x_{(0100)}\left(t_1^{-2}t_2y\right)\right) = \left|t_1^2t_2^{-1}\right|_F^{-5s} f_s\left(w_2x_{(0100)}\left(t_1^{-2}t_2y\right)\right) .
\end{aligned}$$

Plugging this into the integral in F_s results in

$$\begin{aligned}
F_s(g) &= |t_1|_F^{5s}\left|t_1^{-1}t_2\right|_F\left|t_1^2t_2^{-1}\right|_F^{-5s}\int_F f_s\left(w_2x_{(0100)}\left(t_1^{-2}t_2y\right)\right)\psi\left(t_1^{-1}t_2y\right)dy = \\
&=\left\{r := t_1^{-2}t_2y\right\} = \left|t_1^{-1}t_2\right|_F\left|t_1t_2^{-1}\right|_F^{-5s}\left|t_1^2t_2^{-1}\right|_F\int_F f_s\left(w_2x_{(0100)}(r)\right)\psi(t_1r)\,dr = \\
&= |t_1|_F\left|t_1t_2^{-1}\right|_F^{-5s}\int_F f_s\left(w_2x_{(0100)}(r)\right)\psi(t_1r)\,dr = \\
&= |t_1|_F\left|t_1t_2^{-1}\right|_F^{-5s}\frac{\zeta(5s-1)}{\zeta(5s)}\left(1-|t_1|_F^{5s-1}q^{-(5s-1)}\right) .
\end{aligned}$$

Case III: Assuming that $g = (t_1,t_2)\,x_\alpha(d)$, we may assume by Proposition 6.1 that

$$|t_1|_F, \left|\frac{t_2}{t_1}\right|_F, |dt_1|_F, \left|dt_1+\frac{t_2}{t_1}\right|_F, \left|\frac{d^2t_1}{4}+\frac{dt_2}{2t_1}\right|_F \le 1$$
$$|d|_F > 1 .$$

We denote $p = \frac{t_1^2}{t_2}d^2 + d$. So

$$\begin{aligned}
F_s(g) &= \int_F f_s\left(w_2w_3x_{-\alpha_1}(1)x_{\alpha+\beta}(r)\,g\right)\psi(r)\,dr = \\
&= \int_F f_s\left(w_2w_3x_{-\alpha_1}(1)x_{(0110)}(r)\,(t_1,t_2)\,x_\alpha(d)\right)\psi(r)\,dr = \\
&= \int_F f_s\left(w_2w_3x_{-\alpha_1}(1)x_{(0110)}(r)\,(t_1,t_2)\,x_{(1000)}(d)\,x_{(0010)}(d)\right)\psi(r)\,dr .
\end{aligned}$$

Here we did not list $x_{(1100)}$,$x_{(0101)}$ or $x_{(0001)}$ since they commute with x_α and $x_{\alpha+\beta}$ and they remain in U under the action of w_2w_3 as shown in previous cases. We then get

$$F_s(g) = \int_F f_s\left(w_2w_3x_{-\alpha_1}(1)x_{(0110)}(r)(t_1,t_2)\,x_{(1000)}(d)\,x_{(0010)}(d)\right)\psi(r)\,dr =$$
$$= \int_F f_s\left((t_1,t_2)^{w_2w_3}\,w_2w_3x_{-\alpha_1}\left(\frac{t_1^2}{t_2}\right)x_{(0110)}\left(\frac{t_1}{t_2}r\right)x_{(1000)}(d)\,x_{(0010)}(d)\right)\psi(r)\,dr\,.$$

We perform a change of variables $r' = \frac{t_1}{t_2}r$ and get

$$F_s(g) = |t_1|_F^{5s}\left|\frac{t_2}{t_1}\right|_F\int_F f_s\left(w_2w_3x_{-\alpha_1}\left(\frac{t_1^2}{t_2}\right)x_{(0110)}(r')\,x_{(1000)}(d)\,x_{(0010)}(d)\right)\psi\left(\frac{t_2}{t_1}r'\right)dr' =$$
$$= |t_1|_F^{5s}\left|\frac{t_2}{t_1}\right|_F\int_F f_s\left(w_2x_{-\alpha_1}\left(\frac{t_1^2}{t_2}\right)x_{(0100)}(r')\,x_{(1000)}(d)\,x_{-(0010)}(d)\right)\psi\left(\frac{t_2}{t_1}r'\right)dr' =$$
$$= |t_1|_F^{5s}\left|\frac{t_2}{t_1}\right|_F\int_F f_s\left(w_2x_{-\alpha_1}\left(\frac{t_1^2}{t_2}\right)x_{(0100)}(r')\,x_{(1000)}(d)\,x_{(0010)}\left(d^{-1}\right)(0010)^\vee\left(d^{-1}\right)k\right)\cdot$$
$$\cdot\,\psi\left(\frac{t_2}{t_1}r'\right)dr'\,,$$

with $k \in K$. We change notations to r instead of r' and get

$$F_s(g) = |t_1|_F^{5s}\left|\frac{t_2}{t_1}\right|_F\int_F f_s\left(w_2x_{-\alpha_1}\left(\frac{t_1^2}{t_2}\right)x_{(0100)}(r)\,x_{(1000)}(d)\,(0010)^\vee\left(d^{-1}\right)\right)\psi\left(\frac{t_2}{t_1}r\right)dr =$$
$$= |t_1|_F^{5s}\left|\frac{t_2}{t_1}\right|_F\int_F f_s\left((0010)^\vee\left(d^{-1}\right)^{w_2}w_2x_{-\alpha_1}\left(\frac{t_1^2}{t_2}\right)x_{(0100)}\left(d^{-1}r\right)x_{(1000)}(d)\right)\psi\left(\frac{t_2}{t_1}r\right)dr =$$
$$= |t_1|_F^{5s}\left|\frac{t_2}{t_1}\right|_F|d|_F^{-5s}\int_F f_s\left(w_2x_{-\alpha_1}\left(\frac{t_1^2}{t_2}\right)x_{(0100)}\left(d^{-1}r\right)x_{(1000)}(d)\right)\psi\left(\frac{t_2}{t_1}r\right)dr =$$
$$= |t_1|_F^{5s}\left|\frac{t_2}{t_1}\right|_F|d|_F^{-5s}\int_F f_s\left(w_2x_{-\alpha_1}\left(\frac{t_1^2}{t_2}\right)x_{(0100)}\left(d^{-1}r\right)\alpha_1^\vee(d)\,x_{-\alpha_1}(d)\,k\right)\psi\left(\frac{t_2}{t_1}r\right)dr$$

with $k \in K$, therefore

$$F_s(g) = |t_1|_F^{5s}\left|\frac{t_2}{t_1}\right|_F|d|_F^{-5s}\int_F f_s\left(\alpha_1^\vee(d)^{w_2}\,w_2x_{-\alpha_1}\left(d^2\frac{t_1^2}{t_2}\right)x_{(0100)}(r)\,x_{-\alpha_1}(d)\right)\psi\left(\frac{t_2}{t_1}r\right)dr =$$
$$= |t_1|_F^{5s}\left|\frac{t_2}{t_1}\right|_F\int_F f_s\left(w_2x_{-\alpha_1}\left(d^2\frac{t_1^2}{t_2}+d\right)x_{(0100)}(r)\right)\psi\left(\frac{t_2}{t_1}r\right)dr =$$
$$= |t_1|_F^{5s}\left|\frac{t_2}{t_1}\right|_F\int_F f_s\left(w_2x_{-\alpha_1}(p)\,x_{(0100)}(r)\right)\psi\left(\frac{t_2}{t_1}r\right)dr\,.$$

We now consider two options:

Option I: Assume $\left|p\right|_{\mathrm{F}} \leq 1$ then

$$\begin{aligned} F_s(g) &= |t_1|_{\mathrm{F}}^{5s} \left|\frac{t_2}{t_1}\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(w_2 x_{(0100)}(r)\, x_{-\alpha_1}(p)\right) \psi\left(\frac{t_2}{t_1} r\right) dr = \\ &= |t_1|_{\mathrm{F}}^{5s} \left|\frac{t_2}{t_1}\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(w_2 x_{(0100)}(r)\right) \psi\left(\frac{t_2}{t_1} r\right) dr = \\ &= \frac{\zeta(5s-1)}{\zeta(5s)} |t_2|_{\mathrm{F}} \left(|t_1|_{\mathrm{F}}^{5s-1} - |t_2|_{\mathrm{F}}^{5s-1} q^{-(5s-1)}\right) . \end{aligned}$$

One should note that this does not depend on d.

Option II: Assume $\left|p\right|_{\mathrm{F}} > 1$ then

$$\begin{aligned} F_s(g) &= |t_1|_{\mathrm{F}}^{5s} \left|\frac{t_2}{t_1}\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(w_2 x_{(0100)}(r)\right) x_{-\alpha_1}(p)\, \psi\left(\frac{t_2}{t_1} r\right) dr = \\ &= |t_1|_{\mathrm{F}}^{5s} \left|\frac{t_2}{t_1}\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(w_2 x_{(0100)}(r)\, x_{\alpha_1}\left(p^{-1}\right) \alpha_1^{\vee}\left(p^{-1}\right) k\right) \psi\left(\frac{t_2}{t_1} r\right) dr \end{aligned}$$

with $k \in \mathrm{F}$. Therefore

$$\begin{aligned} F_s(g) &= |t_1|_{\mathrm{F}}^{5s} \left|\frac{t_2}{t_1}\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(w_2 x_{(1100)}\left(-rp^{-1}\right) x_{\alpha_1}\left(p^{-1}\right) x_{\alpha_2}(r)\, \alpha_1^{\vee}\left(p^{-1}\right)\right) \psi\left(\frac{t_2}{t_1} r\right) dr = \\ &= |t_1|_{\mathrm{F}}^{5s} \left|\frac{t_2}{t_1}\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(x_{\alpha_1}\left(-rp^{-1}\right) x_{(1100)}\left(p^{-1}\right) w_2 x_{\alpha_2}(r)\, \alpha_1^{\vee}\left(p^{-1}\right)\right) \psi\left(\frac{t_2}{t_1} r\right) dr = \\ &= |t_1|_{\mathrm{F}}^{5s} \left|\frac{t_2}{t_1}\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(w_2 x_{\alpha_2}(r)\, \alpha_1^{\vee}\left(p^{-1}\right)\right) \psi\left(\frac{t_2}{t_1} r\right) dr = \\ &= |t_1|_{\mathrm{F}}^{5s} \left|\frac{t_2}{t_1}\right|_{\mathrm{F}} \int_{\mathrm{F}} f_s\left(\alpha_1^{\vee}\left(p^{-1}\right)^{w_2} w_2 x_{\alpha_2}\left(p^{-1} r\right)\right) \psi\left(\frac{t_2}{t_1} r\right) dr = \\ &= |t_1|_{\mathrm{F}}^{5s} \left|\frac{t_2}{t_1}\right|_{\mathrm{F}} |p|_{\mathrm{F}}^{-5s} \int_{\mathrm{F}} f_s\left(w_2 x_{\alpha_2}\left(p^{-1} r\right)\right) \psi\left(\frac{t_2}{t_1} r\right) dr . \end{aligned}$$

With a change of variables $r' = p^{-1} r$ we get

$$F_s(g) = |t_1|_{\mathrm{F}}^{5s} \left|\frac{t_2}{t_1}\right|_{\mathrm{F}} |p|_{\mathrm{F}}^{-5s+1} \int_{\mathrm{F}} f_s\left(w_2 x_{\alpha_2}(r')\right) \psi\left(\frac{t_2}{t_1} p r'\right) dr' .$$

If $\left|\frac{t_2}{t_1} p\right|_{\mathrm{F}} \leq 1$ then

$$F_s(g) = \frac{\zeta(5s-1)}{\zeta(5s)} |t_2|_{\mathrm{F}} \left(\left|t_1 p^{-1}\right|_{\mathrm{F}}^{5s-1} - |t_2|_{\mathrm{F}}^{5s-1} q^{-(5s-1)}\right) .$$

If on the other hand $\left|\frac{t_2}{t_1}p\right|_{\mathrm{F}} > 1$, denote $l := \frac{t_2}{t_1}p$ and $q^a = |l|_{\mathrm{F}}$ and then

$$\begin{aligned}
F_s(g) &= |t_1|_{\mathrm{F}}^{5s}\left|\frac{t_2}{t_1}\right|_{\mathrm{F}}|p|_{\mathrm{F}}^{-5s+1}\int_{\mathrm{F}} f_s\left(w_2 x_{\alpha_2}(r)\right)\psi\left(\frac{t_2}{t_1}pr\right)dr = \\
&= \int_{\mathcal{O}} \psi l r dr + \int_{\mathrm{F}\setminus\mathcal{O}} f_s\left(w_2 x_{\alpha_2}(r)\right)\psi(lr)\,dr = \\
&= |l|_{\mathrm{F}}^{-1}\int_{l\mathcal{O}} dr' + \sum_{k=1+a}^{\infty} q^{-5ks}\int_{\varpi^{-k}\mathcal{O}} \psi(lr)\,dr = 0\,.
\end{aligned}$$

The integrals vanish since $\varpi^{-1}\mathcal{O} \subset l\mathcal{O}$ and ψ is of conductor $\mathcal{O}$.

□

7.2 Computing $F_s * P_s$

In the following section we use the identification between roots of ${}^L\mathrm{G}(\mathbb{C})$ and coroots of $\mathrm{G}(\mathrm{F})$ as described in chapter 2. In particular, we identify the first fundamental weight ω_1 of ${}^L\mathrm{G}$ with $\alpha^\vee + 2\beta^\vee$.

We remember that $\mathcal{O}/(\varpi) = \mathbb{F}_q$ and that $\left|\mathcal{O}/(\varpi)^2\right| = q^2$. We fix a set of representatives A in $\mathcal{O}$ for $\mathbb{F}_q$ and a set of representatives B in $\mathcal{O}$ for $\mathcal{O}/(\varpi)^2$.

In order to calculate this convolution, we first need to know more about $A_{[0,0]}$ and $A_{[1,0]}$. By [Gro98] we know that

$$\begin{aligned}
A_{[0,0]} &= 1_{\mathrm{K}} \\
A_{[1,0]} &= q^{-3}\left(1_{\mathrm{K}[1,0]\mathrm{K}} + 1_{\mathrm{K}}\right)
\end{aligned}$$

Convolution with 1_{K} does not change a function in $C\left((\mathrm{U},\psi)\setminus\mathrm{G}/\mathrm{K}\right)$ so we need to calculate what the convolution with $1_{\mathrm{K}[1,0]\mathrm{K}} = 1_{\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}}$ equals. By

Iwasawa decomposition $\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K} = \coprod_i \mathrm{K}\,b_i$ where $b_i \in \mathrm{B}$. For $g \in \mathrm{G}$

$$\begin{aligned}
(F_s * 1_{\mathrm{K}[1,0]\mathrm{K}})(g) &= \int_{\mathrm{G}} F_s\left(gh^{-1}\right) 1_{\mathrm{F}[1,0]\mathrm{F}}(h)\,dh = \\
&= \int_{\mathrm{K}[1,0]\mathrm{K}} F_s\left(gh^{-1}\right) dh = \\
&= \sum_i \int_{\mathrm{K}\,b_i} F_s\left(gh^{-1}\right) dh = \\
&= \sum_i \int_{\mathrm{K}} F_s\left(gb_i^{-1}k^{-1}\right) dk = \\
&= \sum_i F_s(gb_i) \ .
\end{aligned}$$

In order to be able to calculate the convolution we need to know how many cosets there are in $\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K} = \coprod_i \mathrm{K}\,b_i$, and find a set of representatives for them. We recall from chapter 1 the number of such cosets, it is equal to

$$\begin{aligned}
|\mathrm{K} \setminus \mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}| &= \mathrm{meas}\,(\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}) = \\
&= \frac{1 + 2q^{-1} + 2q^{-2} + 2q^{-3} + 2q^{-4} + 2q^{-5} + q^{-6}}{1 + q^{-1}} q^6 = \\
&= q + q^2 + q^3 + q^4 + q^5 + q^6 \ .
\end{aligned}$$

7.2.1 Description of Left Cosets

The results in this section are taken from [GGS02]. They give a description of the decomposition of $\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}$ into left cosets of the form $ul\,\mathrm{K}$ with $u \in \mathrm{U}$ and $l \in \mathrm{M}$. We say that $l_1 \equiv l_2$ if $l_1 \in \mathrm{M}(\mathcal{O})\,l_2\,\mathrm{M}(\mathcal{O})$. The first result counts these cosets:

Proposition 7.2 ([GGS02] Proposition 13.3)**.** *In the decomposition of* $\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}$*,* *the number of distinct cosets of the form* $ul\mathrm{K}$ *is given by*

$$\begin{cases} 1, & l \equiv -\omega_1(\varpi) \\ q(q+1), & l \equiv -\beta^\vee(\varpi) \\ q^4(q+1), & l \equiv (\alpha^\vee+\beta^\vee)(\varpi) \\ q^6, & l \equiv \omega_1(\varpi) \\ q^3-1, & l \equiv 1 \\ 0, & \text{otherwise} \end{cases}.$$

For $l \not\equiv 1$ the following proposition gives a list of representatives for the left cosets:

Proposition 7.3 ([GGS02]Proposition 14.2)**.** *If* l *lies in the double coset of either*

$$-(3\alpha+2\beta)^\vee(\varpi),\ -\beta^\vee(\varpi),\ (3\alpha+\beta)^\vee(\varpi),\ (3\alpha+2\beta)^\vee(\varpi)$$

then representatives of the distinct left cosets of $\mathrm{U}(\mathcal{O}) \cap l\,\mathrm{U}(\mathcal{O})\,l^{-1}$ *in* $\mathrm{U}(\mathcal{O})$ *give the distinct left cosets of the form* $ul\mathrm{K}$ *in* $\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}$.

In order to exhaust the list of such left cosets we need to construct the cosets with $l \equiv 1$. This construction is described in the following result:

Let $\mathrm{U}(\mathcal{O}) \subset \mathrm{U}^* \subset \mathrm{U}(\mathrm{F})$ be the group obtained from $\mathrm{U}(\mathcal{O})$ by adjoining the central elements $x_{3\alpha+2\beta}\left(\frac{t}{\varpi}\right)$ with $t \in \mathcal{O}$. In fact, every element $u \in \mathrm{U}^*$ can be written as $ux_{3\alpha+2\beta}\left(\frac{t}{\varpi}\right)$ with $u \in \mathrm{U}(\mathcal{O})$ and $t \in \mathcal{O}^\times$. We denote by U_2 the root subgroup of $3\alpha+2\beta$. $V_1(\mathcal{O}) = \mathrm{U}(\mathcal{O})\,/\,\mathrm{U}_2(\mathcal{O})$ is free of rank 4 over $\mathcal{O}$. Let

$$m \subset \frac{1}{\varpi}V_1(\mathcal{O})\,/V_1(\mathcal{O})$$

be a line stable under a Borel subgroup of $\mathrm{L}(\mathcal{O}/(\varpi)) = \mathrm{GL}_2(\mathcal{O}/(\varpi))$. This type of line is called a *singular line*. We have $m = \theta(l)$ where l is a line in $(\mathcal{O}/(\varpi))^2$ and

$$\theta(x,y) = x^3e_\beta + x^2ye_{\alpha+\beta} + xy^2e_{2\alpha+\beta} + y^3e_{3\alpha+\beta}\,,$$

where $\left\{e_\gamma\right\}$ is a Chevalley basis normalized by

$$\left[e_\beta, e_\alpha\right] = e_{\alpha+\beta},\ \left[e_\beta, e_{\alpha+\beta}\right] = 2e_{2\alpha+\beta},\ \left[e_\beta, e_{2\alpha+\beta}\right] = 3e_{3\alpha+\beta}\,.$$

Our Chevalley base is a little different and corresponds to

$$f_\alpha = e_\alpha,\ f_\beta = e_\beta,\ f_{\alpha+\beta} = \frac{1}{2}e_{\alpha+\beta},\ f_{2\alpha+\beta} = \frac{1}{4}e_{2\alpha+\beta},\ f_{3\alpha+\beta} = -\frac{3}{2}e_{3\alpha+\beta},\ f_{3\alpha+2\beta} = \frac{1}{8}e_{3\alpha+2\beta}$$

Since all the coefficients here are invertible in $\mathcal{O}$, this makes no difference.

Let $V_1(m)$ be the corresponding $\mathcal{O}$-module between $\frac{1}{\varpi}V_1(\mathcal{O})$ and $V_1(\mathcal{O})$ and let $\mathrm{U}(m)$ be the subgroup of $\mathrm{U}(\mathrm{F})$ with

$$\mathrm{U}(m) \cap \mathrm{U}_2(\mathrm{F}) = \frac{1}{\varpi}\mathrm{U}_2(\mathcal{O}), \quad \mathrm{U}(m) / (\mathrm{U}(m) \cap \mathrm{U}_2(\mathrm{F})) = V_1(m)\ .$$

We have $\mathrm{U}(\mathcal{O}) \subset \mathrm{U}^* \subset \mathrm{U}(m)$ with $[\mathrm{U}(m) : \mathrm{U}(\mathcal{O})] = q^2$ and the $q+1$ subgroups $\mathrm{U}(m)$ of U intersect in U^*. The exhaustion of all left cosets is given by:

Proposition 7.4 ([GGS02] Proposition 14.5)**.** *If $u \in \mathrm{U}(m) \setminus \mathrm{U}(\mathcal{O})$ then $u\mathrm{K} \subset \mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}$. The representatives u of the q^2-1 non-trivial cosets of $\mathrm{U}(\mathcal{O})$ in $\mathrm{U}(m)$ give distinct single cosets $u\mathrm{K}$. As we vary the lines m in $\frac{1}{\varpi}V_1(\mathcal{O})/V_1(\mathcal{O})$, we obtain the remaining q^3-1 distinct cosets of $\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}$.*

Remark 7.1. For any two lines m_1, m_2 in the above proposition and $u_1 \in \mathrm{U}(m_1)$, $u_2 \in \mathrm{U}(m_2)$ we may have $u_1\mathrm{K} = u_2\mathrm{K}$ only if $u_1, u_2 \in \mathrm{U}^*$.

7.2.2 Explicit Construction of Right Cosets

In this section we give an explicit construction of a set of representatives of right cosets in the decomposition of $\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}$.

We will now construct the left cosests of K in $\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}$. We list here a set of representatives of all left cosets in this decomposition constructed using the above-mentioned results:

- For $l \equiv -\omega_1(\varpi)$ we have the following representative:

$$\{-\omega_1(\varpi)\}\ .$$

- For $l \equiv -\beta^\vee(\varpi)$ we have the following representatives:

$$\{u(r_1,r_2,r_3,r_4,r_5)\,\omega_1(\varpi) | r_1,r_2,r_3,r_4 \in A,\ r_5 \in B\}\ .$$

- For $l \equiv (\alpha^\vee + \beta^\vee)(\varpi)$ we have the following representatives:

$$\{u(r_1,0,0,0,0)(-\alpha^\vee - \beta^\vee)(\varpi) \big| r_1 \in A\} \cup$$
$$\cup \{u(0,0,0,r_4,0)\,x_\alpha(s)(-\beta^\vee)(\varpi) \big| r_4, s \in A\}\ .$$

- For $l \equiv \omega_1(\varpi)$ we have the following representatives:

$$\{u(r_1,r_2,0,0,r_5)(\beta^\vee)(\varpi) \big| r_2,r_5 \in A,\ r_1 \in B\} \cup$$
$$\cup \{u(0,0,r_3,r_4,r_5)\,x_\alpha(s)(\alpha^\vee + \beta^\vee)(\varpi) \big| r_3,r_5,s \in A, r_4 \in B\}\ .$$

- For $l \equiv 1$ we have the following representatives:

$$\left\{x_{3\alpha+2\beta}\left(\frac{r_5}{\varpi}\right) \middle| r_5 \in A,\ r_5 \not\equiv 0\right\} \cup$$
$$\cup \left\{x_\beta\left(\frac{r_1}{\varpi}\right) x_{3\alpha+2\beta}\left(\frac{r_5}{\varpi}\right) \middle| r_1,r_5 \in A,\ r_1 \not\equiv 0\right\} \cup$$
$$\cup \left\{x_\beta\left(\frac{y^3 r_1}{\varpi}\right) x_{\alpha+\beta}\left(\frac{y^2 r_1}{\varpi}\right) x_{2\alpha+\beta}\left(\frac{y r_1}{\varpi}\right) x_{3\alpha+\beta}\left(\frac{r_1}{\varpi}\right) x_{3\alpha+2\beta}\left(\frac{r_5}{\varpi}\right) \middle| r_1,r_5,y \in A,\ r_1 \not\equiv 0\right\}\ .$$

Remark 7.2. The above calculations are stated for left cosets of the form $ul\mathrm{K}$ with l toral and $u \in \mathrm{U}$. In our calculation we need right cosets. We will now describe how to get such representatives. Let $w = \left(w_\alpha w_\beta\right)^3$ be the longest element in the Weyl group of G_2. For any $\gamma \in \Phi$ and $\gamma^\vee \in \Phi^\vee$ we have $w(\gamma) = -\gamma$ and $w(\gamma^\vee) = -\gamma^\vee$. So w acts on unipotent elements as transpose and on toral elements as inversion.

From the above, we can write

$$\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K} = \coprod_i b_i\,\mathrm{K} = \coprod_i u_i n_i z_i\,\mathrm{K}\,,$$

with $u_i \in \mathrm{U}$, $z_i \in \mathrm{T}$ and $n_i = x_\alpha(a_i)$ where $a_i \in \mathrm{F}$. It is clear that we can also write

$$\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K} = \coprod_i w u_i n_i z_i\,\mathrm{K}\ .$$

On the other hand,

$$\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K} = {}^t(\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}) = {}^t\left(\coprod_i w u_i n_i z_i \mathrm{K}\right) = \coprod_i \cdot \mathrm{K}\,{}^t z_i \cdot {}^t n_i \cdot {}^t u_i \cdot {}^t w =$$
$$= \coprod_i \mathrm{K}\, z_i \cdot {}^t n_i \cdot {}^t u_i w = \coprod_i \mathrm{K}\, z_i^w \cdot {}^t n_i^w \cdot {}^t u_i^w = \coprod_i \mathrm{K}\, z_i^{-1} n_i u_i \,.$$

7.2.3 Computing the Convolution

Using the results from previous sections we are able to calculate

$$F_s * (P_1 A_{[0,0]} + P_2 A_{[1,0]})$$

at $g = (t_1, t_2)\, x_\alpha(d)$. Computing $F_s * \mathbb{1}_{\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}}$ yields

$$(F_s * \mathbb{1}_{\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}})(g) = \int_{\mathrm{G}} F_s\left(gh^{-1}\right) \mathbb{1}_{\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}}(h)\, dh = \int_{\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}} F_s\left(gh^{-1}\right) dh =$$
$$= \sum_i \int_{\mathrm{K}\,{}^t b_i^w} F_s\left(gh^{-1}\right) dh = \left\{h = k\,{}^t b_i^w,\ dh = dk\right\} =$$
$$= \sum_i \int_{\mathrm{K}} F_s\left(g\left({}^t b_i^w\right)^{-1} k^{-1}\right) dk = \sum_i F_s\left(g\left({}^t b_i^w\right)^{-1}\right) =$$
$$= \sum_i F_s\left(g u_i^{-1} n_i^{-1} z_i\right) = \sum_i F_s\left(\left(u_i^{-1}\right)^g g n_i^{-1} z_i\right) =$$
$$= \sum_i \psi\left(\left(u_i^{-1}\right)^g\right) F_s\left((t_1, t_2)\, z_i x_\alpha\left(\alpha(z_i)(d - a_i)\right)\right) \,.$$

If we denote $z_i = (x_i, y_i)$ then we can write

$$(F_s * \mathbb{1}_{\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}})(g) = \sum_i \psi\left(\left(u_i^{-1}\right)^g\right) F_s\left((t_1 x_i, t_2 y_i)\, x_\alpha\left(\frac{y_i}{x_i^2}(d - a_i)\right)\right) \,.$$

And finally we get

$$F_s * \left(P_1\left(q^{-(5s-2)}\right) A_{[0,0]} - P_2\left(q^{-(5s-2)}\right) A_{[1,0]}\right)(g) =$$
$$= \left(P_1\left(q^{-(5s-2)}\right) - q^{-3} P_2\left(q^{-(5s-2)}\right) F_s(g)\right) +$$
$$+ q^{-3} P_2\left(q^{-(5s-2)}\right)\left(\sum_i \psi\left(\left(u_i^{-1}\right)^g\right) F_s\left((t_1 x_i, t_2 y_i)\, x_\alpha\left(\frac{y_i}{x_i^2}(d - a_i)\right)\right)\right) \,.$$

In the course of calculating this sums we need the following Gaussian sums

Lemma 7.2. *For an additive character ψ on* F *with conductor $\mathcal{O}$ we have*

$$\sum_{\substack{r\in A\\ r\neq 0}} \psi\left(z\frac{r}{\varpi}\right) = \begin{cases} q-1, & |z|_{\mathrm{F}} < 1 \\ -1, & |z|_{\mathrm{F}} = 1 \\ 0, & |z|_{\mathrm{F}} > 1 \end{cases}$$

Proof. The case of $|z|_{\mathrm{F}} < 1$ is clear. Assume that $|z|_{\mathrm{F}} = 1$ by making a change of representatives for A we may assume that $z = 1$ in which case

$$-1 = \int_{\varpi^{-1}\mathcal{O}^\times} = \left\{ \begin{matrix} t = \frac{u}{\varpi} \\ u \in \mathcal{O} \\ dt = q du \end{matrix} \right\} = \int_{\mathcal{O}^\times} \psi\left(\frac{u}{\varpi}\right) q du = q \sum_{\substack{r\in A\\ r\neq 0}} \int_{r+(\varpi)} \psi\left(\frac{u}{\varpi}\right) du =$$
$$= q \sum_{\substack{r\in A\\ r\neq 0}} \int_{(\varpi)} \psi\left(\frac{r+p}{\varpi}\right) dp = q \sum_{\substack{r\in A\\ r\neq 0}} \psi\left(\frac{r}{\varpi}\right) \int_{(\varpi)} \psi\left(\frac{p}{\varpi}\right) dp = \sum_{\substack{r\in A\\ r\neq 0}} \psi\left(\frac{r}{\varpi}\right)$$

The case of $|z|_{\mathrm{F}} > 1$ is similar. □

Remark 7.3. For $y \in A$ we have $\left|y^2 + y\right|_{\mathrm{F}} < 1$ if and only if $y^2 + y \in \varpi$ which is equivalent to $\varpi | (y^2 + y) \subset (y+1)(y)$ as ideals and this equivalent to $y \equiv 0$ or $y \equiv -1$.

It remains to prove the equality

$$D_s{}^{\psi} = j(s) F_s * P_s \ . \tag{7.1}$$

We shall validate this equality at the identity element of G_2:

Proposition 7.5.

$$D_s{}^{\psi}(e) = j(s)(F_s * P_s)(e) \ .$$

Proof. By the above computations we have

$$
\begin{aligned}
&(F_s * \mathbb{1}_{\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}})(e) = F_s\left(\left(\frac{1}{\varpi},\frac{1}{\varpi^2}\right)\right) + \sum_{r_1,r_2,r_3,r_4\in A,r_5\in B} \psi\left(u(r_1,r_2,r_3,r_4,r_5)^{-1}\right) F_s\left(\varpi,\varpi^2\right) + \\
&+ \sum_{r_1\in A} F_s\left(\left(\frac{1}{\varpi},\frac{1}{\varpi}\right)\right) + \sum_{r_4,s\in A} F_s\left(\left(1,\frac{1}{\varpi}\right)x_\alpha\left(-\frac{s}{\varpi}\right)\right) + \\
&+ \sum_{r_2,r_5\in A,r_1\in B} F_s((1,\varpi)) + \sum_{r_3,r_5,s\in A,r_4\in B} F_s\left((\varpi,\varpi)\,x_\alpha\left(-\frac{s}{\varpi}\right)\right) + \\
&+ \sum_{r_5\in A,r_5\neq 0} F_s(1) + \sum_{r_1,r_5\in A,r_1\neq 0} F_s(1) + \sum_{r_1,r_2,y\in A,r_1\neq 0} \psi\left(-\frac{y^2+y}{\varpi}r_1\right) F_s(1) = \\
&= 0 + \frac{q^{6-10s}+q^{6-5s}}{\zeta(5s)} + 0 + 0 + \frac{q^{4-5s}}{\zeta(5s)} + \frac{2q^{4-5s}}{\zeta(5s)} + \frac{q-1}{\zeta(5s)} + \frac{q^2-q}{\zeta(5s)} + \frac{q^2}{\zeta(5s)} = \\
&= \frac{q^{6-10s}+q^{5-5s}+3q^{4-5s}+2q^2-1}{\zeta(5s)}.
\end{aligned}
$$

Therefore $j(s)(F_s * P_s)(e)$ equals

$$
\begin{aligned}
&j(s)\left(P_1\left(q^{-(5s-2)}\right) - q^{-3}P_2\left(q^{-(5s-2)}\right)\right)F_s(e) + q^{-3}P_2\left(q^{-(5s-2)}\right)(F_s * \mathbb{1}_{\mathrm{K}\,\omega_1(\varpi)\,\mathrm{K}})(e) = \\
&= \frac{\zeta(5s)\,\zeta(5s-1)^2\,\zeta(10s-4)}{\zeta(5s+1)\,\zeta(5s-1)\,\zeta(5s-2)} \cdot \\
&\cdot \frac{P_1\left(q^{-(5s-2)}\right) - q^{-3}P_2\left(q^{-(5s-2)}\right) - q^{-3}P_2\left(q^{-(5s-2)}\right)\left(q^{6-10s}+q^{5-5s}+3q^{4-5s}+2q^2-1\right)}{\zeta(5s)}.
\end{aligned}
$$

Simplifying this expression yields

$$
j(s)(F_s * P_s)(e) = \frac{1+2q^{1-5s}}{\zeta(5s+1)} = D_s{}^{\psi}(e) \quad !
$$

This is an equality of polynomials in q^{-s}. □

After this thesis was submitted eq. (7.1) was proven for all $g \in \mathrm{G}$ and hence Conjecture 4.2 was proved.

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