Research Statement - Avner Segal

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My research lies in the crossroad between number theory and representation theory. In particular, I am interested in automorphic representations, the representation theory of p-adic and Lie groups and their application to number theory. This document consists of two parts. Part 1 is a short introduction to the Langlands program and a survey of my past projects. In Part 2, I lay out current projects and future plans.

Part 1. Introduction and Previous Work

1. Introduction

The theory of automorphic representations developed as a branch of modern number theory extending the classical theory of automorphic forms. The theory of automorphic representations is the theory of non-Abelian Fourier analysis on an adelic algebraic group.

The Langlands program ([41]) is one of the driving forces in the theory of automorphic representations. By and large, the Langlands program draws connections between irreducible automorphic representations of algebraic groups[1] and the study of representations of the absolute Galois group of a finite extension F of $\mathbb{Q}$. The Langlands conjectures for the group GL_1 are equivalent to *class field theory* ([71]).

We denote the ring of adeles of F by $\mathbb{A}_F$; it is the restricted product of the completions F_ν where ν are all the primes of F including ∞. In order to give a more accurate description of the Langlands conjectures, I shall assume in this section that $F=\mathbb{Q}$ and consider what the Langlands conjectures say for a split group G. In this case, an automorphic representation of G is defined to be a subquotient of the space of functions on $G(\mathbb{A}_\mathbb{Q})$ which are left-$G(\mathbb{Q})$-invariant and satisfy some finiteness conditions. These finiteness conditions essentially require that these functions have moderate growth at cusps, are a solution of a system of differential equations whose image under Hecke operators is a finite linear-combination of Hecke-eigenforms. Such functions are called *automorphic forms on* $G(\mathbb{A}_\mathbb{Q})$.

Langlands constructed an associated complex group ${}^LG$ called *the Langlands dual group* using the root system dual to that of G [2]. Another ingredient in the conjecture is the conjectural Langlands group $\mathcal{L}_\mathbb{Q}$; this group is conjectured to be a cover by a compact group of the Weil group $W_\mathbb{Q}$ of $\mathbb{Q}$. The Weil group $W_\mathbb{Q}$ is a dense subgroup of the absolute Galois group of $\mathbb{Q}$. A conjectural construction for $\mathcal{L}_\mathbb{Q}$ is given in [1].

The Langlands conjectures asserts the existence of a functorial bijection between the following sets:

- **The Galois side:** The set of equivalence classes of continuous homomorphisms $\phi:\mathcal{L}_\mathbb{Q}\to {}^LG$, called *Langlands parameters.*
- **The automorphic side:** Certain sets of irreducible automorphic representations called *global L-packets.*

One of the features of the conjectured bijection is that it should preserve $\mathcal{L}$-functions, which are complex functions that can be associated both to Langlands parameters (Artin $\mathcal{L}$-functions) and to irreducible automorphic representations.

Currently, there is no known general construction for the $\mathcal{L}$-function of irreducible automorphic representations but its analytic properties are closely related to those of partial $\mathcal{L}$-functions. Let π be an irreducible automorphic representation; π factors as a restricted tensor product $\otimes_{p\leq\infty}\pi_p$ of irreducible representations π_p of $G(\mathbb{Q}_p)$ such that almost all of these representations are spherical, i.e. admit a non-zero fixed-vector fixed by $G(\mathbb{Z}_p)$. For an irreducible spherical representation π_p, with $p<\infty$, the Satake isomorphism associates a semi-simple conjugacy class $t_{\pi_p}\in {}^LG$. For a finite dimensional representation ρ of ${}^LG$, Langlands defined the local $\mathcal{L}$-function at the prime p to be

$$\mathcal{L}_p(s,\pi_p,\rho)=\left(det\left(1-\rho\left(t_{\pi_p}\right)p^{-s}\right)\right)^{-1}\quad\forall s\in\mathbb{C}. \tag{1}$$

For a finite set of primes S such that $\infty\in S$ and π_p is unramified for any $p\notin S$, the partial $\mathcal{L}$-function is defined to be

$$\mathcal{L}^S(s,\pi,\rho)=\prod_{p\notin S}\mathcal{L}_p(s,\pi_p,\rho) \tag{2}$$

The idea at the heart of Langlands program is functoriality. Let G and H be two algebraic groups with a map $\iota:{}^LG\to{}^LH$. For any Langlands parameter $\phi:\mathcal{L}_\mathbb{Q}\to{}^LG$ for G, we get a Langlands parameter $\iota\circ\phi:\mathcal{L}_\mathbb{Q}\to{}^LH$ for H. Assuming the Langlands conjectures, this would imply a map sending irreducible automorphic representations of G to irreducible automorphic representations of H preserving $\mathcal{L}$-functions. In fact, one of the ways to detect images of functorial lifts is through the analytic behaviour of certain $\mathcal{L}$-functions.

[1] One also considers finite central covers of such groups.

[2] In the non-split case, the Langlands dual group contains a similar complex group as a normal subgroup

The main example of such results is Rallis' far-reaching project connecting the analytic behaviour of $\mathcal{L}$-functions of cuspidal representations of a rank n orthogonal group with the non-vanishing of their θ-lift to a rank m metaplectic group.

One method for studying the analytic behaviour of $\mathcal{L}$-functions, and functoriality as a consequence, is the Rankin-Selberg method (see [6] for a comprehensive survey and [16, 22, 36] for a few notable examples). The idea is to represent the $\mathcal{L}$-function using an integral whose analytic properties are known. The method is, in a sense, a generalization of Riemann's proof of the analytic continuation of the Riemann zeta-function [59].

Many such integral representations are constructed using the Rankin-Selberg method demonstrated below. Let G be a subgroup of H. Let π be a cuspidal representation of G and let $\mathcal{E}(s,f,g)$ (here $s \in \mathbb{C}$, f is a section of some degenerate principal series of H and $g \in H(\mathbb{A}_{\mathbb{Q}})$) be an Eisenstein series on H. An integral representation is an equality of the form

$$\int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})} \varphi(g)\,\mathcal{E}(s,f,g)\,dg = \mathcal{L}^S(s,\pi,\rho)\,d(s,f_S,\varphi)\,, \tag{3}$$

where $\varphi \in \pi$ and $d(s,\varphi)$ is a meromorphic function determined by φ and the Eisenstein series such that for any $s_0 \in \mathbb{C}$, data can be chosen so that it is holomorphic and non-vanishing at s_0. The meromorphic continuation of $\mathcal{L}^S(s,\pi,\rho)$ then follows. Furthermore, the orders of the poles of $\mathcal{L}^S(s,\pi,\rho)$ are bounded by those of $\mathcal{E}(s,f,g)$.

Any Rankin-Selberg construction (such as Equation (3)) usually requires two main steps: one is a process often called *unfolding* and the other is performing a local computation. The process of unfolding rewrites the integral on the left-hand side in terms of factorizable data, namely both the integrand and the domain of integration should be restricted products (both $G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})$ and $\mathcal{E}(s,f,g)$ are **not** restricted products). The unfolding is followed by proving a local identity between the local $\mathcal{L}$-factor and a local integral, over a p-adic group, determined by Theorem 1.

A special kind of Rankin-Selberg integral is an integral which, when unfolded, contains a non-unique model (in the sense of [8]). Such integrals are called *new-way integrals*, named after the title of [53]. Such integrals prove to be more difficult to deal with since, due to the lack of uniqueness, a stronger local identity is required in order to show that the unfolded integral is Eulerian (for some examples of such integrals, see [8, 9, 32, 55, 56, 66]).

Another application of an equality such as Equation (3) is in the study of functorial lifts. Assume that $\mathcal{L}^S(s,\pi,\rho)$ admits a pole of the same order as $\mathcal{E}(s,f,g)$ at $s=s_0$ and the residual representation of $\mathcal{E}(s,f,g)$ at $s=s_0$ is small, in the sense of [7]. Then, the small representation method can be used to describe the image of π under corresponding functorial lifts. A key example of this can be seen in the study of theta-correspondence via the Siegel-Weil identity. This project, led by S. Rallis, was the topic of many works; [57] survey this project.

In my research I apply techniques from various fields in mathematics such as functional analysis (in particular C^*-algebras), algebraic geometry, motivic integration and application of computer programs.

2. Previous Work

The main object of my M.Sc. and Ph.D. ([65]) theses was to study a family of integral representations for the standard twisted partial $\mathcal{L}$-function for cuspidal representations of the exceptional group of type G_2. In what follows, I present parts of the results. For a more detailed account, see [32, 66].

For a number field F, let $G = G_2 \times GL_1$, where G_2 denotes the exceptional group of this type defined over F. For any cuspidal representation π of $G(\mathbb{A}_F)$ we fix a quasi-split form H of $Spin_8$ and a normalized degenerate Eisenstein series $\mathcal{E}(s,f,g)$ along the Heisenberg parabolic subgroup of H. The following is an integral representation of the standard $\mathcal{L}$-function of π in the sense of Equation (3).

Theorem 1 ([32, 66]). *For any $\varphi \in \pi$ and a finite set S of places of F such that all data is unramified outside of S, it holds that:*

$$\int_{G_2(F)\backslash G_2(\mathbb{A}_F)} \mathcal{E}(s,f,g)\,\varphi(g)\,dg = \mathcal{L}^S\left(s+\frac{1}{2},\pi,\mathfrak{st}\right) d_S(s,\varphi_S,f_S) \tag{4}$$

is an integral representation of $\mathcal{L}^S(s,\pi,\mathfrak{st})$.

This is a new-way integral in the sense of [53], as explained above, and therefore the local theory developed is more involved. Another novelty in the proof is the notion of an *approximation to a generating function*. Any proof of a Rankin-Selberg integral representation relies on a local unramified calculation which in turn relies on a generating series for the local $\mathcal{L}$-function. In particular, these calculations rely on a closed-form formula for the generating series. While working on [32, 66], we did not have such a closed-form formula. As an alternative, we introduced an approximation to the generating series in the sense of [66, pg. 31]. This method requires us to view it as a limit of elements in the reduced C^*-algebra of G. To the best of my knowledge, this technique had not been used before in the study of Rankin-Selberg integrals.

In order to study the analytic behaviour of $\mathcal{L}^S(s,\pi,\mathfrak{st})$ and its connection to functorial lifts of π, I studied the analytic behaviour of the various Eisenstein series $\mathcal{E}(s,f,g)$ in Theorem 1.

Theorem 2. *The Eisenstein series $\mathcal{E}(s,f,g)$ along the Heisenberg parabolic subgroup of a quasi split-form H of $Spin_8$ are holomorphic on the half-plane $Re(s)>0$ except for possible poles at $s_0=\frac{1}{2}$, $s_0=\frac{3}{2}$ and $s_0=\frac{5}{2}$. All poles are at most simple except for the case studied in [19, Proposition 9.1] where the pole is at most double.*

A more precise statement can be found in [63, Theorem 4.1]. This theorem proves [26, Conjecture 1.1] when $|S|$ is large enough. In [64] I obtain a detailed description of the residual representations of these Eisenstein series.

I will now present two applications of Theorem 1 and the results of [63] to the study of cuspidal representations of G with emphasis on exceptional Θ-lifts. The first one is [63, Theorem 7.2]. Once again, in order to simplify notations, I quote only a partial result.

Theorem 3. *For a cuspidal representation π of $G(\mathbb{A})$, the following are equivalent:*

1. *The partial $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,\mathfrak{st})$ admits a pole of order 2 at $s=2$.*
2. *π has a non-trivial lift to $S_3(\mathbb{A})$, where S_3 is the finite-type symmetric group.*

*Furthermore, these are **CAP** (cuspidal associated to parabolic) representations of G with respect to its Borel subgroup.*

Another application of these theorems is the topic of [31]. There we prove the following result characterizing Rallis-Schiffmann lifts (previously studied in [58] and [18]) via the analytic behaviour of $\mathcal{L}^S(s,\pi,st)$:

Theorem 4. *For a cuspidal irreducible representation π of $G_2(\mathbb{A})$ the following are equivalent*

1. *There exists an automorphic irreducible square integrable representation τ of $SO(2,1)(\mathbb{A})$ such that π is a weak lift of the representation $\tau\boxtimes\mathbf{1}$ of $SO(2,1)\times SO(2,1)$ with respect to the map r.*
2. *The partial $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,st)$ has a pole at $s=2$.*

The pole of the $\mathcal{L}$-function is simple unless π is a weak lift of $\mathbf{1}\boxtimes\mathbf{1}$ in which case the pole is of order 2.

In the course of the proof of this theorem we made use of a Siegel-Weil-like identity ([31, Theorem 3.8]) between the leading terms of two Eisenstein series. The two other main ingredients of the proof are a see-saw duality and a regularization of the theta lift of principal series representations of $\widetilde{SL_2}$ to representations of a quasi-split special-orthogonal group of an 8-dimensional space.

Part 2. Present and Future Projects

3. Representation Theory of G_2 and Exceptional Theta Lifts

As explained above, the orders of poles of $\mathcal{E}\left(s-\frac{1}{2},f,g\right)$ provide a bound on the orders of poles of $\mathcal{L}(s,\pi,\mathfrak{st})$. It is easy to show that $\mathcal{L}^S(s,\pi,\mathfrak{st})$ (note the shift by $\frac{1}{2}$ here) is holomorphic at $s=3$. As for $s=1$ and $s=2$ we conjecture that the bounds in Theorem 2 are tight, namely:

Conjecture 1. *For any Eisenstein series $\mathcal{E}(s,f,g)$ as in Theorem 1, which admits a pole of order n at $s_0\in\{1,2\}$, there exists a cuspidal representation π of $G(\mathbb{A})$ associated to it such that $\mathcal{L}^S(s,\pi,\mathfrak{st})$ admits a pole of order n at s_0.*

One approach to prove Conjecture 1 is, given s_0 as above, characterizing all cuspidal representations of $G(\mathbb{A})$ such that $\mathcal{L}^S(s,\pi,\mathfrak{st})$ admits a pole at s_0 in terms of functorial lifts. This would be a natural application of Theorem 1, Theorem 2 and the results of [64]. This is done in the case of $s_0=2$ as described in Theorem 3 and Theorem 4.

Project 1. *Characterize all cuspidal representations of $G(\mathbb{A})$ such that $\mathcal{L}^S(s,\pi,\mathfrak{st})$ admits a pole at $s_0=1$ in terms of functorial lifts.*

To be more concrete, it is conjectured that:

Conjecture 2. *If $\mathcal{L}(s,\pi,\mathfrak{st})$ admits a pole at $s=1$ (it is automatically a simple pole) then there exists a form $\mathfrak{G}$ of PGL_3 such that π has a non-trivial (exceptional) theta lift to $\mathfrak{G}$.*

Note that the case of π generic with $\pi|_{GL_1}=\mathbf{1}$ follows from [25] and [27]. The case where $\pi|_{GL_1}=\mathbf{1}$ is an ongoing project of Wee Teck Gan and Gordan Savin. It seems that a strategy similar to that of [31], combined with results from [64] and [21], can supply a proof for this conjecture.

4. Square Integrable Spectrum and Degenerate Residual Spectrum of Exceptional Groups

The study of the automorphic spectrum of a reductive group can be broken up into the study of the cuspidal spectrum and the residual spectrum of the group. While constructing the cuspidal spectrum often requires application of various tools such as the trace formula, functorial lifts etc., the residual spectrum is constructed using cuspidal representations of Levi subgroups by the theory of Eisenstein series.

The square-integrable residual spectra of simple groups of relative rank 2 have been computed in a series of works such as [29, 39, 40, 45, 72]. The only case missing is the residual spectrum of non-quasi-split groups with absolute root system E_6 and relative root system of type G_2. These groups, E_6^D, are parametrized by cubic division algebras D.

Project 2. *Compute the residual spectrum of the group* $E_6^D(\mathbb{A})$.

Relatively little is known about the residual spectrum of groups with relative rank higher than 2. For classical groups, the square-integrable residual spectrum with cuspidal data along the Borel has been computed in [47, 48]. For exceptional groups this was done only for the group of type G_2.

The key challenge in dealing with other exceptional groups lies in their ranks and the size of their Weyl groups. One approach to describing parts of the residual spectrum (with cuspidal data along the Borel subgroup) is to study the degenerate residual spectrum, i.e. the part of the spectrum arising from parabolic induction from one-dimensional representations of maximal Levi subgroups. Such a study in the case of the groups GL_n and Sp_{2n} is described in [**?**, 33, 34].

Project 3. *Compute the degenerate residual spectrum of split exceptional groups of type* E_n ($n = 6, 7, 8$) *along maximal parabolic subgroups.*

Computing the degenerate residual spectrum is a more feasible project than computing the full residual spectrum for various reasons such as:

- The constant term formula (which controls the combinatorics of the computation) is given by a sum of intertwining operators. The number of summands in the sum is the size of a quotient of Weyl groups which (in most cases) is significantly smaller than the size of the original Weyl group.
- The space of unramified characters of a maximal Levi subgroup is 1-dimensional, which allows us to deal with functions of one complex variable.

As can be seen in Section 2, degenerate Eisenstein series and their residues are interesting objects since:

- Degenerate Eisenstein series appear in many Rankin-Selberg constructions, e.g. [26, 35, 52, 53, 55, 56, 66].
- Degenerate residual representations (not necessarily square-integrable) play important roles in constructions of minimal and small representations and computations of theta lifts (see [7, 19, 31, 42] for examples).

5. Structure of Degenerate Principal Series Representations of Exceptional Groups

During the course of my work on [63, 64], I had to address a few questions regarding the structure of local degenerate principal series representations of quasi-split forms of $Spin_8$. Specifically, the question of reducibility of these representations, the identification of their maximal semi-simple quotients and their images under normalized intertwining operators. In fact, the structure theory of degenerate principal series representations is, on its own, an interesting problem in the representation theory of p-adic groups. This question has been discussed in [2, 37, 38] for classical groups and in [13, 14, 49] for split exceptional groups of type G_2 and F_4.

A key ingredient of Project 3 is the study of the structure of local degenerate principal series. The following is a joint project with H. Halawi, who is a graduate student at Ben-Gurion University.

Project 4. *Study reducibility and structure of degenerate principal series induced from maximal parabolic subgroups of exceptional groups of type* E_n ($n = 6, 7, 8$).

In order to do this, we have created a computer program, which implements ideas presented in [10, 14, 68] and computes reducibility points and gathers information on the structure of reducible degenerate principal series. As of now, the analysis of degenerate principal series of E_6 is essentially complete and we are currently studying the degenerate principal series of E_7.

6. Basic Functions and Generating Functions of $\mathcal{L}$-functions

As explained in Part 1, all Rankin-Selberg constructions require a proof of an equality between the local $\mathcal{L}$-factor and an integral over a p-adic group determined by the global integral. This equality, usually, requires a closed-form formula for the generating function of the local $\mathcal{L}$-factor. In this section I introduce the generating function and the associated *basic function*; I then outline a few fascinating problems related to them. For a more detailed introduction, the reader is referred to [51] and [46].

For simplicity, let $G = \mathbf{G}(\mathbb{Q}_p)$ be a split p-adic group and let $K = \mathbf{G}(\mathbb{Z}_p)$ be a maximal compact subgroup. Also, let ${}^LG$ denote its (complex) Langlands dual group.

Given an irreducible unramified representation π of G (namely π contains non-zero K-fixed vectors), let $t_\pi \in {}^LG$ denote a representative of the semi-simple conjugacy-class associated to π by the Satake isomorphism (see [30] for more details). For any irreducible finite-dimensional representation ρ of ${}^LG$, the local $\mathcal{L}$-factor of π is given in Equation (1).

The *generating function* Δ_s of $\mathcal{L}(s,\pi,\rho)$ is the image of $\mathcal{L}(s,\pi,\rho)$ under the *inverse Satake transform*. Namely, for any unramified representation π of G, any non-zero spherical $v^0 \in \pi$ and any $l \in \pi^*$, Δ_s the following identity is satisfied:

$$\int_G \Delta_s(g)\, l\left(\pi(g)\, v^0\right) dg = \mathcal{L}(s,\pi,\rho)\, l\left(v^0\right). \tag{5}$$

We are then able to describe Δ_s in terms of the so called *spectral basis* of the spherical Hecke algebra $\mathcal{H} = C_c^\infty(K\backslash G/K)$. This basis is parametrized by the finite-dimensional irreducible representations of ${}^LG$. The coefficients of the transformation matrix between the spectral basis and the geometric basis (comprised of the characteristic functions of sets of the form KgK) are closely connected to the affine Kazhdan-Lusztig polynomials. For more details, see [30, 32, 46, 51].

The *basic function* associated to $\mathcal{L}(s,\pi,\rho)$ is $\Phi = \lim_{s\to 0} \Delta_s$ when it is well defined. Note that Δ_s and Φ depend only on ρ, not on π. The generating functions and basic functions of local $\mathcal{L}$-factors are interesting for various reasons, such as:

- As mentioned, they play a role in the proof of equalities such as Equation (3).
- They can be used in order to define the local $\mathcal{L}$-factor for ramified representations á la Godement-Jacquet theory (see [36]).
- In the context of *beyond endoscopy* ([44]), substitution of basic functions into the relative trace formula is believed to play an important role in describing functorial lifts. For more details see [61].

However, for most of these applications, a closed-form formula for Φ is required with the exception of the first one as described after Theorem 1. A few notable works discussing these functions are [3–5, 11, 23, 24, 43, 46, 51, 60, 61]. The common goal in all of these works is to describe Φ as an eigenfunction of a Fourier transform acting on the space of Schwartz functions on the Vinberg monoid associated to G.

If $G = \mathbf{G}(\mathbb{F}_p[[t]])$ is equipped with a determinant form $det : G \to \mathbb{G}_m$ such that the kernel of det is semi-simple, then [3, Theorem 4.1] gives an elegant description of the generating and basic function of $\mathcal{L}(s,\pi,\rho)$ in terms of the trace of the Frobenius operator on the arc space of the Vinberg monoid X. This construction, however, cannot be repeated for characteristic zero.

I will now propose a few projects connected to the generating function of $\mathcal{L}(s,\pi,\rho)$ and its application to Rankin-Selberg integrals. For what follows, assume that G is the $\mathbb{Q}_p$-points of a group with a determinant form such that the kernel is semi-simple, as in [46, Section 1].

6.1. **Approximations of the Generating Function.** As shown in [32, 55], it can be enough to use an approximation of the generating function in order to prove the required local identity in the Rankin-Selberg method. An approximation of Δ_s is another function $D_s \in \mathcal{H}[[p^{-s}]]$ such that there exists $P_s \in \mathcal{H}$ such that

$$D_s = \Delta_s * P_s \tag{6}$$

and $*P_s$ is an invertible operator for $Re(s) \gg 0$.

An example for such approximations in the case of $G = GSp_4$, with ρ the spin representation, is discussed in [62, Appendix 1], [67] and [46, Subsec. 6.2]. One considers the monoid $X = MSp_4$ given by the Zariski closure of GSp_4 in $M_4(\mathbb{Q}_p)$; the determinant form on G is the similitude factor μ. Taking Φ to be the characteristic function of $M_4(\mathbb{Z}_p)$ and $D_s(g) = \Phi(g)\,\mu(g)^{s+\frac{3}{2}}$ yields an equality such as Equation (6), where $P_s \in \mathcal{H}[p^{-s}]$ is given explicitly in [67].

Let ρ be a quasi-miniscule representation of ${}^LG$ and let λ_ρ be the highest weight of ρ. In particular, λ_ρ is a dominant co-weight of G. Let Λ be the W-invariant sublattice of $X^*(T)$ generated by λ_ρ. By Cartan decomposition, $G = KT^+K$, any bi-K invariant function is defined by its values on $T^+/(T^+\cap K) = \Lambda^+$. Let

$$D_s(\lambda(\varpi)) = \begin{cases} p^{-(\sum k_i)s}, & \lambda = \sum k_i (w_i\lambda_\rho), \forall k_i \geq 0 \\ 0, & otherwise \end{cases}.$$

This formulation, which is based on several know examples, was communicated to me by A. Pollack.

Conjecture 3. *If we denote by Δ_s the generating function associated with the local $\mathcal{L}$-factor $\mathcal{L}(s,\cdot,\rho)$, then there exists $P_s \in \mathcal{H}[p^{-s}]$ so that Equation (6) holds.*

This is a generalization of a conjecture of G. Shimura ([67]) for the case $G = GSp_{2n}$ and the spin $\mathcal{L}$-function. In [54], A. Pollack proves this for the standard $\mathcal{L}$-function of a quasi-split $G = GSpin(V)$. It is also known for the spin $\mathcal{L}$-function of GSp_{2n} for $n = 1, 2, 3$.

6.2. **The Casselman-Shalika Formula.** Another attempt to overcome the lack of a general formula for Δ_s in terms of the geometric basis of $\mathcal{H}$ is by describing various transforms of Δ_s, such as Fourier transforms along unipotent subgroups; for example, the Whittaker-Fourier coefficient. In fact, most Rankin-Selberg constructions boil down to proving an identity such as

$$\mathcal{W}(\Delta_s) = F_s, \tag{7}$$

where $\mathcal{W}(\Delta_s)$ is a transform of Δ_s and F_s is an explicit function, where both $\mathcal{W}$ and F_s are determined by the Rankin-Selberg integral. For simplicity, I will address only the Fourier coefficients of Δ_s along unipotent subgroups. Let U be a standard unipotent subgroup of G and Ψ a character of U. For any $l \in \mathrm{Hom}_U(\pi, \Psi)$, Fubini's theorem implies that

$$\mathcal{L}(s, \pi, \rho)\, l\left(v^0\right) = \int\limits_{U\backslash G} \Delta_s^{(U,\Psi)}(g)\, l(\pi(g) v_0)\, dg,$$

where $f^{(U,\Psi)}$ denotes the following Fourier transform of f:

$$f^{(U,\Psi)}(g) = \int\limits_{U} f(ug)\, \overline{\Psi(u)} du.$$

However, as already mentioned, the generating function only has an abstract construction which makes it difficult to evaluate $\Delta^{(U,\Psi)}$. In the case where G is split, $U = N$ is a maximal unipotent subgroup and Ψ is a non-degenerate character of N one can use a variant of the Casselman-Shalika formula introduced in [17, Theorem 5.2]. This variant gives a simple form for the Whittaker-Fourier coefficient of the spectral basis of $\mathcal{H}$.

Since Δ_s is given in terms of the spectral basis, this variant of the Casselman-Shalika formula, together with the decomposition of the symmetric algebra of ρ, provides a conceptual way to perform the unramified calculation to Rankin-Selberg integrals that unfold with a Whittaker model.

In this method we made two important assumptions: that G is split and that N is maximal. One would like to remove these assumptions. It seems like the first assumption can be weakened using similar methods to those implied in [20, 69].

Project 5. *Prove the Casselman-Shalika formula for quasi-split groups.*

Removing the assumption that N is maximal, is a far more challenging task. I suggest an approach which incorporates both the recent achievements described in [3] and ideas from motivic integration.

6.3. **Transfer From Positive Characteristic.** The study of motivic integration was suggested by M. Kontsevich in 1995. This is a branch of algebraic geometry and model theory that assigns a "volume" to subsets of arc spaces of algebraic varieties; these volumes are elements of the Grothendieck ring of algebraic varieties. In particular, the field of motivic harmonic analysis applies this theory to the study of harmonic analysis of reductive groups over local non-Archimedean fields. See [15] for a detailed introduction.

Motivic harmonic analysis can be used for various applications such as the study of uniform bounds on orbital integrals or the transfer of results between of positive characteristic and that of characteristic zero.

For example, J. L. Waldspurger's work ([70]) on transfer factors and endoscopic matching for characteristic zero was transferred by J. Gordon and T. Hales ([28]) to characteristic p, where p is a large enough prime.

These transfer principles are the starting point for the following attempt to evaluate $\Delta_s^{(U,\Psi)}$.

Project 6 (Joint with J. Gordon)**.**

- *Consider the generating function Δ_s constructed in [3] (only for local fields with **positive characteristic**) and show that it is a definable function in the sense of Denef-Pas language ([15]).*
- *Find a closed-form formula for $\Delta_s^{(U,\Psi)}$ in the setting of [3].*
- *Use the above to transfer this formula to the characteristic zero case.*

This project brought me to learn new techniques coming from algebraic geometry and motivic integration.

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