Rankin-Selberg Integrals with a Non-Unique Model for the Standard $\mathcal{L}$-function of Cuspidal Representations of the Exceptional Group of Type G_2

Thesis Submitted in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

By

Avner Segal

Submitted to the Senate of Ben-Gurion University of the Negev

September 10, 2015

Beer Sheva

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Approved by the advisor: ______________
Approved by the Dean of the Kreitman School of Advanced Graduate Studies: ______________

September 10, 2015

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This work was carried out under the supervision of Dr. Nadya Gurevich
In the Department of Mathematics
Faculty of Natural Sciences

Research-Student's Affidavit when Submitting the Doctoral Thesis for Judgment

I, Avner Segal, whose signature appears below, hereby declare that (Please mark the appropriate statements):

X I have written this Thesis by myself, except for the help and guidance offered by my Thesis Advisors.

X The scientific materials included in this Thesis are products of my own research, culled from the period during which I was a research student.

__ This Thesis incorporates research materials produced in cooperation with others, excluding the technical help commonly received during experimental work. Therefore, I am attaching another affidavit stating the contributions made by myself and the other participants in this research, which has been approved by them and submitted with their approval.

Date: ____________ Student's name: ____________ Signature:____________

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In memory of my grandfather Hans Meir
who taught me that in order to get far,
you need to be both quick and durable.

'Well, there's...' Colon racked his brains. 'There's algebra. That's like sums with letters. For... for people whose brains aren't clever enough for numbers, see?'

Terry Pratchett, Jingo

Contents

Abstract

Let π be a cuspidal representation of the simple, split exceptional group of type G_2, χ be a Hecke character of $F^\times \backslash \mathbb{A}^\times$ and $\mathfrak{st} : G_2(\mathbb{C}) \to GL_7(\mathbb{C})$ be the standard embedding.

The main objective of this thesis is to prove the meromorphic continuation of the standard twisted $\mathcal{L}$-function $\mathcal{L}(s, \pi, \chi, \mathfrak{st}) = \mathcal{L}(s, \pi \boxtimes \chi, \mathfrak{st})$ of π. We achieve this by attaching to $\mathcal{L}(s, \pi, \chi, \mathfrak{st})$ an integral representation that involves a cusp form in π and a degenerate Eisenstein series. This integral representation binds the analytic properties of $\mathcal{L}(s, \pi, \chi, \mathfrak{st})$ and those of the Eisenstein series.

The integral in question unfolds with a non-unique model. A *new way* method, named after [35], reduces the unramified computation to an identity that involves the generating function of $\mathcal{L}(s, \pi, \chi, \mathfrak{st})$. The complexity of the generating function constitutes the main complication in this computation. We introduce the notion of an *approximation to the generating function* and use this in order to perform the local unramified computation.

The Eisenstein series in the integral is the one associated with a degenerate principal series induced from the Heisenberg parabolic subgroup of a quasi-split form of $Spin_8$. We study the poles of this Eisenstein series in the half-plane $\mathfrak{Re}(s) > 0$ in order to find bounds to the orders of poles of $\mathcal{L}(s, \pi, \chi, \mathfrak{st})$.

We further use the bounds on the orders of poles of $\mathcal{L}(s, \pi, \chi, \mathfrak{st})$ in order to classify all **CAP** representation of G_2 with respect to the Borel subgroup in terms of the analytic behavior of $\mathcal{L}(s, \pi, \chi, \mathfrak{st})$. We prove that these representations are precisely the ones arising as functorial lifts from a finite-type group.

Key words: Automorphic forms, automorphic $\mathcal{L}$-functions, integral representations, new way integrals, Eisenstein series, Langlands program.

Introduction

$\mathcal{L}$-functions are one of the central objects of modern number theory, representation theory, the theory of automorphic forms and other areas of mathematics. They are used in order to acquire algebraic and arithmetic information by means of analytic inquiries. The use of $\mathcal{L}$-functions in number theory goes back to Dirichlet who used them in 1837 [9] to prove that any arithmetic progression with coprime coefficients contains infinitely many prime numbers. This thesis deals with automorphic $\mathcal{L}$-functions.

Let F be a number field and let $\mathcal{P}$ be its set of places. For any $\nu \in \mathcal{P}$ we denote by F_ν the completion of F at ν. We also denote by $\mathbb{A}_F = \mathbb{A}$ the ring of adeles of F.

Let G be a split reductive group defined over F and let ${}^LG$ be its complex dual Langlands group. Given a non-Archimedean place ν of F and an unramified representation π_ν of $G(F_\nu)$, there is a semisimple element $t_{\pi_\nu} \in {}^LG$ determined up to conjugacy, called the Satake parameter of π_ν. For a finite dimensional representation ρ of ${}^LG$ the local $\mathcal{L}$-factor is defined by

$$\mathcal{L}(s, \pi_\nu, \rho) = \frac{1}{\det\left(I - \rho\left(t_{\pi_\nu}\right) q_\nu^{-s}\right)},$$

where q_ν is the cardinality of the residue field of F_ν.

Given an irreducible automorphic representation $\pi = \otimes_\nu \pi_\nu$ of $G(\mathbb{A}_F)$ and a finite set of places S, such that $\nu \not| \infty$ and π_ν is unramified for $\nu \notin S$, the global partial $\mathcal{L}$-function is defined by

$$\mathcal{L}^S(s, \pi, \rho) = \prod_{\nu \notin S} \mathcal{L}(s, \pi_\nu, \rho).$$

This product converges for $Re(s) \gg 0$ and R.P. Langlands conjectured that it admits a meromorphic continuation to the whole complex plane. The most effective way known to prove this conjecture, for various cases, is by attaching to $\mathcal{L}^S(s, \pi, \rho)$ an integral representation with convenient analytic properties. One method of doing this is the Rankin-Selberg method. For an excellent survey of the Rankin-Selberg method consult [5].

Let G denote the simple, simply connected and adjoint group of type G_2. Let π be an irreducible cuspidal representation of the exceptional group $G_2(\mathbb{A}_F)$ and let χ be a Hecke character of $GL_1(\mathbb{A}_F)$. The Langlands dual group of G_2 is isomorphic to $G_2(\mathbb{C})$ and the Langlands dual group of GL_1 is isomorphic to $\mathbb{C}^\times$. We denote by $\mathfrak{st}$ the irreducible seven-dimensional complex representation of $G_2(\mathbb{C}) \times \mathbb{C}^\times$. The **main objective** of this thesis is to prove the meromorphic continuation of the partial $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st}) = \mathcal{L}^S(s,\pi \boxtimes \chi,\mathfrak{st})$ by attaching to it an integral representation.

We let $P = M \cdot U$ be the Heisenberg parabolic subgroup of G with Levi part M and unipotent radical U. In Chapter 1 we introduce the following bijection

$$\left\{\begin{array}{c}\text{Isomorphism classes}\\ \text{of quasi-split}\\ \text{forms of } Spin_8\,/F\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Isomorphism classes}\\ \text{of étale cubic}\\ \text{algebras over } F\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Non-degenerate}\\ M(F)\text{-orbits of}\\ \text{characters of } U(\mathbb{A}) \to \mathbb{A}\end{array}\right\}$$

For an étale cubic algebra E over F we denote by H_E a corresponding quasi-split form. We fix a non-trivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ and denote by Ψ_E the composition of a corresponding non-degenerate character $U(\mathbb{A}) \to \mathbb{A}$ and ψ. It has been shown [13, Theorem 3.1] that any cuspidal representation of G_2 supports at least one non-degenerate Fourier coefficient along U corresponding to an étale cubic algebra E over F.

We consider a family of zeta integrals $\mathcal{Z}_E(\chi,s,\varphi,f)$, to be introduced in Chapter 2, parametrized by E as above. The integral $\mathcal{Z}_E(\chi,s,\varphi,f)$ involves a normalized degenerate Eisenstein series $\mathcal{E}^*_E(\chi,s,f,g)$ on H_E associated with the induced representation from the Heisenberg parabolic subgroup P_E of H_E. More precisely, we fix the Heisenberg parabolic subgroup $P_E = M_E \cdot U_E$ with Levi subgroup M_E and unipotent radical U_E. As

$$M_E \cong \left(\operatorname{Res}_{E/F} GL_2\right)^0 = \left\{g \in \operatorname{Res}_{E/F} GL_2 : \det(g) \in F^\times\right\},$$

a determinant $\det_{M_E}$ is defined. For a Hecke character χ of $F^\times \backslash \mathbb{A}^\times$ we form the unnormalized parabolic induction

$$I_{P_E}(\chi, s) = \mathrm{Ind}_{P_E}^{H_E}\left(\chi \otimes |\cdot|^{s+\frac{5}{2}}\right) \circ \det_{M_E}. \tag{0.1}$$

For a standard section $f_s \in I_{P_E}(\chi, s)$ we define the following Eisenstein series:

$$\mathcal{E}_E(\chi, s, f, g) = \sum_{\gamma \in P_E(F)\backslash H_E(F)} f_s(\gamma g).$$

This series converges for $\mathfrak{Re}(s) \gg 0$ and admits a meromorphic continuation to the whole complex plane. The normalization of $\mathcal{E}_E(\chi, f_s, s, g)$ is introduced in Chapter 1.

The main result of this thesis is

Theorem 2.0.2 . *Let π be an irreducible cuspidal representation of $G_2(\mathbb{A}_F)$ supporting the Fourier coefficient corresponding to E and let $\varphi = \otimes_{\nu\in\mathcal{P}}\varphi_\nu \in \pi$ and $f_s = \otimes_{\nu\in\mathcal{P}} f_{s,\nu} \in I_{P_E}(\chi, s)$ be factorizable data. Fix a finite set of places $S \subset \mathcal{P}$, containing all Archimedean places, so that for $\nu \notin S$ all data is unramified. Let*

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \int\limits_{G_2(F)\backslash G_2(\mathbb{A}_F)} \varphi(g)\, \mathcal{E}_E^*(\chi, s, f, g)\, dg. \tag{0.2}$$

Then

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \mathcal{L}^S\left(s + \frac{1}{2}, \pi, \chi, \mathfrak{st}\right) d_S(\chi, s, \Psi_E, \varphi, f), \tag{0.3}$$

where

$$d_S(\chi, s, \Psi_E, \varphi, f) = \frac{\mathcal{Z}_E(\chi, s, \varphi, f)}{\mathcal{L}^S\left(s + \frac{1}{2}, \pi, \chi, \mathfrak{st}\right)}$$

is a meromorphic function for any $\varphi \in \pi$ and K-finite section f_s. Moreover, for any s_0 there exist vectors φ_S, f_S such that $d_S(\chi, s, \Psi_E, \varphi_S, f_S)$ is analytic in a neighborhood of s_0 and $d_S(\chi, s_0, \Psi_E, \varphi_S, f_S) \neq 0$.

In particular, the family of twisted partial $\mathcal{L}$-functions $\mathcal{L}^S(s, \pi, \chi, \mathfrak{st})$ admits a meromorphic continuation to the whole complex plane.

This result is a generalization of [24, Theorem 3.1] where we constructed an integral representation for the standard (untwisted) $\mathcal{L}$-function $\mathcal{L}^S(s, \pi, \mathfrak{st})$, where π supports the Fourier coefficient associated with the split cubic algebra $F \times F \times F$ over F and χ is the trivial character.

The foremost challenge in proving Theorem 2.0.2 is that in the unfolded integral there appears a non-unique model, namely a functional from $\mathrm{Hom}_{U(\mathbb{A}_F)}(\pi, \mathbb{C}_{\Psi_E})$ which for many π-s is not a one-dimensional space. Such integrals, called *new way integrals* after [35], are not *a priori* factorizable. In [35] I. Piatetski-Shapiro and S. Rallis suggested a method to prove the factorizability of such integrals. The remarkable mechanism suggested there relies on the fact that **for any** functional from $\mathrm{Hom}_{U(F_\nu)}(\pi_\nu, \mathbb{C}_{\Psi_{E,\nu}})$ the local integral at an unramified place ν is equal to the local $\mathcal{L}$-factor multiplied by the valuation of the functional on the spherical vector.

It is worth mentioning that there are only a few known examples (just [6], [7], [24], [35], [37], [38] to the best of the author's knowledge) of Rankin-Selberg integrals that unfold with a non-unique model and have been shown to represent $\mathcal{L}$-functions.

As a byproduct of the proof of Theorem 2.0.2 we provide a parametrization of the $G_2(F)$ orbits in $P_E(F)\backslash H_E(F)$. This parametrization over the algebraic closure $\overline{F}$ of F is particularly nice, as it can be related to the orbits of Möbius transformations on $\mathbb{P}^1(\overline{F}) \times \mathbb{P}^1(\overline{F}) \times \mathbb{P}^1(\overline{F})$.

The proof of Theorem 2.0.2 is mostly unified for all étale cubic algebras E over F and only splits up for the different kinds of cubic algebras at a very late stage. This occurs in Sections 4.2 and 4.4, where the arithmetical difference between the different étale algebras becomes critical.

In the proof of the local unramified identity we use the same strategy as in [24] and use an *approximation to the generating function* of the local $\mathcal{L}$-factor instead of the generating function itself. In Chapter 4 we strengthen the results of [24] concerning this approximation. To the best of our knowledge, this is the first time that this method has been used. It was later used in [37] and [38] as well for $\mathcal{L}$-functions of cuspidal representations of classical groups.

Other Rankin-Selberg integrals representing the standard $\mathcal{L}$-function of cuspidal representations of G_2 were introduced in [36] and [19] for generic representations and in [20] for any cuspidal representation of G_2. The latter is done using a doubling construction showing that the set of poles of $\mathcal{L}(s, \pi, \chi, \mathfrak{st})$ is contained in the set of poles of a degenerate Eisenstein series of the exceptional group of type E_8. D. Ginzburg and J. Hundley further conjectured [20, Conjecture 1] that for $\mathfrak{Re}(s) \geq 0$ the orders of the poles are bounded by 2. This conjecture is settled in Chapter 7.

It is then natural to ask what the possible poles of $\mathcal{L}^S(s, \pi, \chi, \mathfrak{st})$ are. We give a partial answer to this question by analyzing the possible poles of $\mathcal{E}_E^*(\chi, s, f, g)$ in the half-plane $\mathfrak{Re}(s) \geq 0$. We say that $\mathcal{E}_E(\chi, \cdot, s, \cdot)$ admits a pole of order n at s_0 if

$$\sup\left\{\operatorname{ord}_{s=s_0} \mathcal{E}_E(\chi, f_s, s, g) : f_s \in I_{P_E}(\chi, s),\ g \in H_E(\mathbb{A})\right\} = n. \tag{0.4}$$

We say that a Hecke character $\chi : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ is of finite order if there exists some $m \in \mathbb{N}$ such that $\chi^m = \mathbb{1}$.

Let χ_E be as follows:

- If $E = F \times F \times F$ then $\chi_E = \mathbb{1}$.

- If $E = F \times K$, where K is a quadratic field extension of F, then $\chi_E = \chi_K$ is the character attached to K by class field theory (note that K/F is automatically a Galois extension).

- If E is a Galois field extension of F then χ_E is the character attached to E by class field theory.

- If E is a non-Galois field extension of F the character χ_E is not defined.

In Chapter 6 we prove the following theorem:

Theorem 6.2.1 . *For any Hecke character $\chi : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ the Eisenstein series $\mathcal{E}_E(\chi, \cdot, s, \cdot)$ is holomorphic at $\mathfrak{Re}(s) > 0$ with the exception of certain combinations of $s_0 \in \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right\}$ and χ a quadratic character or $\chi = \chi_E$. The following table lists all the possible poles of $\mathcal{E}_E(\chi, s, f, g)$ and gives bounds on their order:*

	$s=\frac{1}{2}$			$s=\frac{3}{2}$		$s=\frac{5}{2}$
	$\chi=\mathbb{1}$	$\chi=\chi_E$	χ quad.	$\chi=\mathbb{1}$	$\chi=\chi_E$	$\chi=\mathbb{1}$
$E=F\times F\times F$	1		1	2		1
$E=F\times K$	1	2	1	1	1	1
E Galois field extension	1	1	1	0	1	1
E non-Galois field extension	1	-	1	0	-	1

Table 1: Bounds on the Order of Poles of $\mathcal{E}_E(\chi, f_s, s, g)$

The proof of the bounds of the poles in Theorem 6.2.1 at $s=\frac{1}{2}$ with $\chi=\mathbb{1}$ and E not a field, relies on a certain conjecture regarding the structure of the degenerate princiapl series at Archimedean places as discussed in Chapter 6. We note that all other results, perhaps with the exception of the proof of [20, Conjecture 1], are independent of this conjecture.

We prove Theorem 6.2.1 by considering the constant term of these Eisenstein series along the Borel subgroup of H_E. The constant term equals a sum of intertwining operators. It is possible that the high-order poles of some of these intertwining operators cancel each other. It is interesting to note that in all cases known to us, cancellation of intertwining operators happens in pairs. In this work, in some instances the cancellations of poles are a little unusual, as they are canceled in triples and quintuples.

As a consequence of Theorem 2.0.2 it holds that

$$ord_{s=s_0}\left(\mathcal{E}_E^*(\chi, f_s, s, g)\right) = ord_{s=s_0}\left(\mathcal{E}_E(\chi, f_s, s, g)\right) \geq ord_{s=s_0}\left(\mathcal{L}^S\left(s+\frac{1}{2}, \pi, \chi, \mathfrak{st}\right)\right). \tag{0.5}$$

This gives a bound on the possible poles of $\mathcal{L}^S(s, \pi, \chi, \mathfrak{st})$ and their orders.

Remark 0.0.1. For any étale cubic algebra E, the residual representation of $\mathcal{E}_E(\mathbb{1}, f_s, s, g)$ at $s_0=\frac{5}{2}l$ is the trivial representation. Also, for a cusp form φ it holds that

$$\int_{G(F)\backslash G(\mathbb{A})} \varphi(g)\, dg = 0.$$

It follows that $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ is holomorphic at $s_0 = 3$.

The following is a natural conjecture:

Conjecture 0.0.2. *For any (E,χ,s_0) with $s_0 \in \left\{\frac{1}{2},\frac{3}{2}\right\}$ such that $\mathcal{E}^*_E(\chi,,s,g)$ admits a pole of order n_0 at s_0 there exists a cuspidal representation π of $G(\mathbb{A})$ supporting the Ψ_E Fourier coefficient such that $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ admits a pole of order n_0 at s_0.*

In Chapter 7 we prove this for the triple $\left(E,\chi_E,\frac{3}{2}\right)$. We apply Theorem 6.2.1 and Equation (0.5) to the study of **CAP** representations of $G_2(\mathbb{A})$ with respect to the Borel. We recall that a cuspidal representation π of G_2 is called a **CAP** (*cuspidal associated with parabolic*) with respect to B if there exists an automorphic character τ of the torus T such that π is nearly equivalent to a subquotient of $\mathrm{Ind}_{B(\mathbb{A})}^{G_2(\mathbb{A})}\tau$.

Let E be an étale cubic algebra which is not a non-Galois field extension. Let χ_E be the character attached to E/F as before and let $S_E = \mathrm{Aut}_F(E)$. We denote by θ_{S_E} the theta lift corresponding to the dual pair $G_2 \times S_E$ in $H_E \rtimes S_E$ studied in [16].

In Chapter 7 we prove the following result:

Theorem 7.1.2 . *Let π be a cuspidal representation of $G(\mathbb{A})$ and let E be an étale cubic algebra over F which is not a non-Galois field extension. The following are equivalent:*

1. *π is a **CAP** representation with respect to B supporting the (U,Ψ_E)-coefficient.*
2. *The partial $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,\chi_E,\mathfrak{st})$ has a pole, of order 2 if $E = F\times F\times F$ or 1 otherwise, at $s=2$.*
3. *The θ-lift $\theta_{S_E}(\pi)$ of π to S_E is non-zero. In particular π is nearly equivalent to the θ-lift $\theta_E(\mathbb{1})$, where $\mathbb{1}$ here is the automorphic trivial representation of $S_E(\mathbb{A})$.*

Following, is a summary of the results in each chapter.

1. In Chapter 1 we introduce the groups at play in this thesis. We also define a few relevant objects, in particular the degenerate Eisenstein series $\mathcal{E}^*_E(\chi,s,f,g)$ taking part in the integral representation of $\mathcal{L}^S\left(s+\frac{1}{2},\pi,\chi,\mathfrak{st}\right)$.

2. In Chapter 2 we define the global zeta-integral $\mathcal{Z}_E(\chi, s, \varphi, f)$ and outline the proof of Theorem 2.0.2. In particular, we derive this theorem from the results of Chapters 3, 4 and 5.

3. In Chapter 3 the unfolding is performed. The most subtle part is parameterizing the space of double cosets $P_E(F)\backslash H_E(F)/G(F)$.

4. In Chapter 4 we perform the local unramified computation. The unramified computation is mostly contained in Chapter 4, though a small portion of the computation is postponed to Appendix A.

5. Chapter 5 is devoted to the ramified computation. We prove that for any s_0 there exist vectors φ_S, f_S such that $d_S(\chi, s, \Psi_E, \varphi_S, f_S)$ is entire and $d_S(\chi, s_0, \Psi_E, \varphi_S, f_S) \neq 0$. The meromorphic continuation of $\mathcal{L}^S(s, \pi, \chi, \mathfrak{st})$ will follow.

6. In Chapter 6 we determine the possible poles for $\mathcal{E}_E(\chi, f_s, s, g)$ in the half-plane $\mathfrak{Re}(s) > 0$. The data needed for some of the computations performed in this chapter can be found in the tables in Appendix B.

7. In Chapter 7 we prove Theorem 7.1.2 regarding the connection between the analytic behavior of $\mathcal{L}^S(s, \pi, \chi_E, \mathfrak{st})$ at $s_0 = 2$, the functorial lift of π to S_E and **CAP** representations.

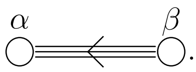
Chapter 1

Preliminaries

Let F be a number field and let $\mathcal{P}$ be its set of places. For any $\nu \in \mathcal{P}$ we denote by F_ν the completion of F at ν. If $\nu < \infty$ we denote by $\mathcal{O}_\nu$ the ring of integers of F_ν, by ϖ_ν a uniformizer of F_ν and by q_ν the cardinality of the residue field of F_ν. We also denote by $\mathbb{A}_F = \mathbb{A}$ the ring of adeles of F.

1.1 The Group G_2

Let G be the simple, split group of type G_2 defined over F. In particular, G is adjoint and simply connected. Let B be a Borel subgroup of G and T a maximal torus in B. Let α and β be the short and long simple roots of G with respect to (B,T). Let W be the Weyl group of G with respect to (B,T). The Dynkin diagram of G is



The set of positive roots of G is

$$\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}.$$

The fundamental weights of G are denoted by

$$\omega_1 = 2\alpha+\beta, \quad \omega_2 = 3\alpha+2\beta.$$

For any simple root γ let $w_\gamma \in W$ be the simple reflection with respect to it. For any root γ we fix a one-parameter subgroup $x_\gamma : \mathbb{G}_a \to G$ as in [28, Section 2]. A realization in terms of these one-parameter subgroups in matrix form is described in Section 4.2. Also, let $h_\gamma : \mathbb{G}_m \to T$ be the coroot subgroup of T such that for any root ϵ it holds that

$$\epsilon\left(h_\gamma\left(t\right)\right) = t^{\langle \epsilon, \gamma^\vee \rangle}.$$

The group G contains a Heisenberg maximal parabolic subgroup $P = M \cdot U$. The Levi subgroup M is isomorphic to GL_2 and is generated by the simple root α, while U is a five-dimensional Heisenberg group. Namely U is a two step unipotent group. Finally, we let $\mathfrak{st} : G \hookrightarrow GL_7$ be the standard embedding of algebraic groups.

1.2 Twisted Partial $\mathcal{L}$-functions

The dual Langlands group ${}^L G$ of G is isomorphic to $G_2\left(\mathbb{C}\right)$.

Let $\pi = \underset{\nu \in \mathcal{P}}{\otimes} \pi_\nu$ be an irreducible cuspidal representation of $G\left(\mathbb{A}\right)$ and let $\chi = \underset{\nu \in \mathcal{P}}{\otimes} \chi_\nu : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be a Hecke character, both unramified outside of a finite subset $S \subset \mathcal{P}$ such that $\mathcal{P}_\infty \subseteq S$. For $\nu \notin S$ we denote the Satake parameter of π_ν by t_{π_ν}. We let

$$\mathcal{L}^S\left(s, \pi, \chi, \mathfrak{st}\right) = \prod_{\nu \notin S} \frac{1}{\det\left(I - \mathfrak{st}\left(t_{\pi_\nu}\right) \chi_\nu\left(\varpi_\nu\right) q_\nu^{-s}\right)}.$$

This product converges for $\mathfrak{Re}\left(s\right) \gg 0$ to an analytic function. Namely, there exists $m > 0$ such that this product converges for $\mathfrak{Re}\left(s\right) > m$ and the limit defines an analytic function on the half-plane $\left\{z \in \mathbb{C} : \mathfrak{Re}\left(z\right) > m\right\} l$. In this thesis we will give a new integral representation which will, among other things, show that $\mathcal{L}^S\left(s, \pi, \chi, \mathfrak{st}\right)$ admits a meromorphic continuation to the whole complex plane.

Remark 1.2.1. The dual Langlands group ${}^L GL_1$ of GL_1 is isomorphic to $GL_1\left(\mathbb{C}\right) = \mathbb{C}^\times$. By abuse of notation we also denote by $\mathfrak{st}$ the irreducible seven-dimensional complex representation of $G_2\left(\mathbb{C}\right) \times \mathbb{C}^\times$. For $\nu \notin S$ the Satake parameter of χ_ν is $\chi_\nu\left(\varpi_\nu\right) \in \mathbb{C}^\times$. It follows that

$$\mathcal{L}^S\left(s, \pi, \chi, \mathfrak{st}\right) = \prod_{\nu \notin S} \frac{1}{\det\left(I - \mathfrak{st}\left(t_{\pi_\nu}, \chi_\nu\left(\varpi_\nu\right)\right) q_\nu^{-s}\right)} = \mathcal{L}^S\left(s, \pi \boxtimes \chi, \mathfrak{st}\right).$$

For χ as above, we let

$$\mathcal{L}^S(s,\chi) = \prod_{\nu \notin S} \frac{1}{1-\chi_\nu(\varpi_\nu) q_\nu^{-s}}. \tag{1.1}$$

More generally, let L be a number field with a set of places $\mathcal{P}_L$ and a ring of adeles $\mathbb{A}_L$. We fix a Hecke character $\chi : L^\times \backslash \mathbb{A}_L^\times \to \mathbb{C}^\times$. For a finite set $S' \subset \mathcal{P}_L$ we let

$$\mathcal{L}_L^{S'}(s,\chi) = \prod_{\nu' \notin S'} \frac{1}{1-\chi_{\nu'}(\varpi_{\nu'}) q_{\nu'}^{-s}}, \tag{1.2}$$

where $\varpi_{\nu'}$ denotes a uniformizer of $L_{\nu'}$ and $q_{\nu'}$ denotes the cardinality of the residue field. Now assume that L is a field extension of F. For a finite set of places $S \subseteq \mathcal{P}$ we let

$$S' = \{\nu' \in \mathcal{P}_L : \nu'|\nu \text{ for } \nu \in S\}.$$

We then note that

$$\mathcal{L}_L^{S'}(s,\chi) = \prod_{\nu \notin S} \left(\prod_{\nu'|\nu} \frac{1}{1-\chi_{\nu'}(\varpi_{\nu'}) q_{\nu'}^{-s}} \right). \tag{1.3}$$

Henceforth we will denote this by $\mathcal{L}_L^S(s,\chi)$. For $\nu \notin S$ we denote

$$\mathcal{L}_{L_\nu}(s,\chi_\nu) = \prod_{\nu'|\nu} \frac{1}{1-\chi_{\nu'}(\varpi_{\nu'}) q_{\nu'}^{-s}}. \tag{1.4}$$

In case that χ_ν is trivial we denote

$$\xi_{L_\nu} = \mathcal{L}_{L_\nu}(s,\mathbb{1}_\nu) = \prod_{\nu'|\nu} \frac{1}{1-q_{\nu'}^{-s}}.$$

1.3 Étale Cubic Algebras over F

We recall that an F-algebra E is said to be *étale*, or *separable*, if $E \otimes_F \overline{F} \cong \overline{F}^n$ for some $n \in \mathbb{N}$. An étale cubic algebra over F is an étale algebra E over F of degree 3. According to [25, eq. (1.8)], E is one of the following:

1. $\underline{E = F \times F \times F}$: This is called the *split cubic algebra* over F. In this case, $\mathrm{Aut}_F(E) \cong S_3$. For $(a,b,c) \in F \times F \times F$ let $\mathrm{Nm}_{E/F}(a,b,c) = abc$.

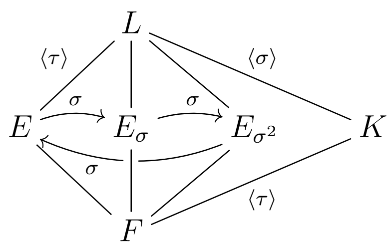
2. $\underline{E = F \times K}$: Here K is a quadratic (and hence Galois) extension of F. Furthermore, $\mathrm{Aut}_F(E) = \mathrm{Gal}(K/F) = \{1, \sigma\}$. For $b \in K$ let $\mathrm{Nm}_{K/F}(b) = bb^\sigma$. Furthermore, for $(a, b) \in F \times K$ let $\mathrm{Nm}_{E/F}(a, b) = a\,\mathrm{Nm}_{K/F}(b) = abb^\sigma$.

3. E is a Cubic Galois Field Extension: In this case, $\mathrm{Aut}_F(E) = \mathrm{Gal}(E/F) = \{1, \sigma, \sigma^2\}$. For $a \in E$ let $\mathrm{Nm}_{E/F}(a) = aa^\sigma a^{\sigma^2}$.

4. E is a Cubic non-Galois Field Extension: In this case, let L be the Galois closure of E over F. This is a sextic Galois extension with

$$\mathrm{Gal}(L/F) = \left\langle \sigma,\ \tau : \sigma^3 = 1,\ \tau^2 = 1,\ (\sigma\tau)^2 = 1 \right\rangle \cong S_3.$$

Note that L is also a Galois extension of E. We achieve the following tower of extensions:



,

where E, $E_\sigma = L^{\langle \sigma\tau\sigma^2 \rangle}$ and $E_{\sigma^2} = L^{\langle \sigma^2\tau\sigma \rangle}$ are the σ-conjugates of E in L. Also, $K = L^{\langle \sigma \rangle}$. For $a \in E$ let $\mathrm{Nm}_{E/F}(a) = aa^\sigma a^{\sigma^2} \in F$.

Remark 1.3.1. We call the first three types **Galois étale cubic algebras over** F.

For reasons that will become clear later in this chapter, we seek a certain realization of E as a quotient of the polynomial ring $F[x]$. Let E be an étale cubic algebra, then one can choose a homogeneous cubic polynomial $p_E \in F[x, y]$ so that $F[x]/(p_E(x, 1)) \cong E$. Note that p_E splits in a finite extension of F into

$$p_E(x, y) = (x - ay)(x - by)(x - cy).$$

In particular, one may write p_E in the form

$$p_E(x, y) = x^3 - T_{(a,b,c)}x^2y + D_{(a,b,c)}xy^2 - N_{(a,b,c)}y^3,$$

where $(a, b, c) \in \overline{F} \times \overline{F} \times \overline{F}$ and

- $T_{(a,b,c)} = a + b + c \in F$.
- $D_{(a,b,c)} = ab + bc + ca \in F$.
- $N_{(a,b,c)} = abc \in F$.

According to [47, Proposition 2.2], for any étale cubic algebra E one can choose a triple (a, b, c) satisfying the following condition:

(CT): (a, b, c) is one of the following:

- $E = F \times F \times F$: $(a, b, c) = (1, -1, 0)$.
- $E = F \times K$, K is a field: Choose $\theta \in K$ such that $K = F[\theta]$ and $\theta + \theta^{\sigma} = 0$. We choose $(a, b, c) = (0, \theta, \theta^{\sigma}) \in K \times K \times K$.
- E is a field: Choose $\theta \in E$ such that $E = F[\theta]$ and $\theta + \theta^{\sigma} + \theta^{\sigma^2} = 0$. We choose $(a, b, c) = \left(\theta, \theta^{\sigma}, \theta^{\sigma^2}\right) \in E \times E_{\sigma} \times E_{\sigma^2}$.

CT stands for *cubic algebra triples.* We henceforth assume that the triple (a, b, c) satisfies **(CT)**. For such triples $T_{(a,b,c)} = 0$.

1.4 Non-degenerate Complex Characters of U

We parametrize the elements of U by

$$u(r_1, r_2, r_3, r_4, r_5) = x_{\beta}(r_1)\, x_{\alpha+\beta}(r_2)\, x_{2\alpha+\beta}(r_3)\, x_{3\alpha+\beta}(r_4)\, x_{3\alpha+2\beta}(r_5).$$

The natural action of M on U induces an action on $\mathrm{Hom}(U, \mathbb{G}_a)$. It is shown in [25] that for any field L of characteristic 0, the $M(L)$-orbits in $\mathrm{Hom}(U(L), L)$ are naturally parametrized by isomorphism classes of cubic L-algebras.

We fix a non-trivial additive complex unitary character $\psi = \underset{\nu \in \mathcal{P}}{\otimes} \psi_{\nu}$ of $F\backslash\mathbb{A}$. This gives rise to injective map

$$\left\{\begin{array}{c}\text{Isomorphism classes of} \\ \text{cubic algebras over } F\end{array}\right\} \hookrightarrow \mathrm{Hom}\left(U(F)\backslash U(\mathbb{A}), \mathbb{C}^{\times}\right)/M(F).$$

In particular, we call $\Psi \in \mathrm{Hom}\left(U\left(F\right)\backslash U\left(\mathbb{A}\right),\mathbb{C}^{\times}\right)/M\left(F\right)$ *non-degenerate* if it is in the image of the étale cubic algebras over F. We then have a correspondence

$$\left\{\begin{array}{c}\text{Isomorphism classes of}\\ \\ \text{étale cubic algebras over } F\end{array}\right\} \longleftrightarrow \mathrm{Hom}^{n.d.}\left(U\left(F\right)\backslash U\left(\mathbb{A}\right),\mathbb{C}^{\times}\right)/M\left(F\right).$$

As in [14], for any étale cubic algebra E over F we will choose a representative Ψ_E of the corresponding $M\left(F\right)$-orbit in $\mathrm{Hom}\left(U\left(\mathbb{A}\right),\mathbb{C}^{\times}\right)$ as follows:

$$\Psi_E\left(u\left(r_1,r_2,r_3,r_4,r_5\right)\right) = \psi\left(r_4 - T_{(a,b,c)}r_3 + D_{(a,b,c)}r_2 - N_{(a,b,c)}r_1\right), \tag{1.5}$$

where (a,b,c) is chosen as in **(CT)**. This character is well defined since

$$\mathcal{Z}\left(U\right) = \left[U,U\right] = \left\{x_{3\alpha+2\beta}\left(r\right) : r \in \mathbb{G}_a\right\},$$

where $\mathcal{Z}\left(U\right)$ is the center of U.

In [24] we use a distinguished representative of the class of complex characters associated with the split cubic algebra over F, denoted by Ψ_s. In Section 3.1 we discuss the relation between Ψ_s and $\Psi_{F\times F\times F}$.

1.5 Fourier Coefficients of a Representation

We denote by $\mathcal{A}\left(G\right)$ the space of automorphic forms on $G\left(\mathbb{A}\right)$. For any $\varphi \in \mathcal{A}\left(G\right)$ and any $\Psi \in \mathrm{Hom}\left(U\left(F\right)\backslash U\left(\mathbb{A}\right),\mathbb{C}^{\times}\right)$ we let

$$L_\Psi\left(\varphi\right)\left(g\right) = \int\limits_{U(F)\backslash U(\mathbb{A})} \varphi\left(ug\right)\overline{\Psi\left(u\right)}du.$$

For any $g \in G\left(\mathbb{A}\right)$ this defines a functional $L_\Psi\left(\cdot\right)\left(g\right) \in \mathrm{Hom}_{U(\mathbb{A})}\left(\mathcal{A}\left(G\right),\mathbb{C}_\Psi\right)$. For an automorphic representation π in $\mathcal{A}\left(G\right)$, we say that π *supports the* (U,Ψ)*-Fourier coefficient* if there exists $\varphi \in \pi$ so that $L_\Psi\left(\varphi\right) \not\equiv 0$. It was shown in [13] that for any cuspidal representation π in $\mathcal{A}\left(G\right)$, there exists an étale cubic algebra E so that π supports the Fourier coefficient corresponding to E. Conversely, it is shown in [16] that for any étale cubic algebra E there exists a cuspidal representation π that supports the (U,Ψ_E)-Fourier coefficient.

1.6 Local Fourier Transform

For a finite $\nu \in \mathcal{P}$, let K_ν denote the maximal compact subgroup $G(\mathcal{O}_\nu)$ of $G(F_\nu)$. Given a complex character Ψ of $U(F_\nu)$ let

$$\mathcal{M}_\Psi = \left\{ f : G(F_\nu) \to \mathbb{C} : f(ugk) = \overline{\Psi(u)} f(g) \ \ \forall u \in U(F_\nu),\ k \in K_\nu \right\}.$$

We let $\mathcal{H}_\nu = \mathcal{H}(G(F_\nu), K_\nu)$ denote the spherical Hecke algebra of $G(F_\nu)$ with respect to K_ν. For $f \in \mathcal{H}_\nu$ define its Ψ-Fourier transform f^Ψ by

$$f^\Psi(g) = \int\limits_{U(F_\nu)} f(ug)\,\Psi(u)\,du. \tag{1.6}$$

Obviously $f^\Psi \in \mathcal{M}_\Psi$.

Remark 1.6.1. For any $f \in \mathcal{M}_\Psi$, f is determined by its values on $M(F_\nu) \cap B(F_\nu)$. For any $g = h_\alpha(t_1) h_\beta(t_2) x_\alpha(d) \in M(F_\nu) \cap B(F_\nu)$ we may also write $g = x_\alpha(p)\, h_\alpha(t_1) h_\beta(t_2)$ with $p = \frac{dt_1^2}{t_2}$. The two different notations will prove useful later.

The following lemma will be useful while evaluating functions in $\mathcal{M}_{\Psi_E}$.

Lemma 1.6.2. *Let E be an étale cubic algebra over F and let (a,b,c) be as in **(CT)**. Assume that $N_{(a,b,c)}, D_{(a,b,c)} \in \mathcal{O}_\nu$. Let $f \in \mathcal{M}_{\Psi_E}$. Then $f(h_\alpha(t_1) h_\beta(t_2) x_\alpha(d)) = 0$ unless the following holds:*

$$N_{(a,b,c)} \frac{t_2^2}{t_1^3} + D_{(a,b,c)} \frac{dt_2}{t_1} + \frac{d^3 t_1^3}{t_2}, D_{(a,b,c)} \frac{t_2}{t_1} + \frac{3d^2 t_1^3}{t_2}, \frac{3dt_1^3}{t_2}, \frac{t_1^3}{t_2} \in \mathcal{O}_\nu. \tag{1.7}$$

Sketch of proof. This lemma follows immediately from the Chevalley relations in G and the fact that

$$f(m) = f(mu) = f\left(\left(mum^{-1}\right) m\right) = \overline{\Psi_E(mum^{-1})} f(m) \quad \forall u \in U(\mathcal{O}_\nu).$$

In particular,

$$\Psi_E\left(mum^{-1}\right) = 1 \quad \forall u \in U(\mathcal{O}_\nu).$$

□

Remark 1.6.3. Note that the condition $N_{(a,b,c)}, D_{(a,b,c)} \in \mathcal{O}_\nu$ in the lemma holds for almost all ν.

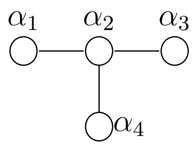
1.7 Quasi-split Forms of D_4

Recall from [43, sec. 3] the following parametrization of quasi-split forms of a split simply-connected algebraic group H defined over F:

$$\{\text{Quasi-split forms of } H \text{ over } F\} \longleftrightarrow \left\{\varphi : \mathrm{Gal}\left(\overline{F}/F\right) \to \mathrm{Aut}\left(Dyn\left(H\right)\right)\right\},$$

where $Dyn\left(H\right)$ is the Dynkin diagram of H.

The Dynkin diagram of type D_4 is given as follows



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We restrict ourselves to the case $H = Spin_8$, the split simply-connected group of type D_4. The quasi-split forms of H were described in [17]. Since $\mathrm{Aut}\left(Dyn\left(Spin_8\right)\right) \cong S_3$ we have

$$\{\text{Quasi-split forms of } Spin_8 \text{ over } F\} \longleftrightarrow \left\{\varphi : \mathrm{Gal}\left(\overline{F}/F\right) \to S_3\right\}.$$

On the other hand, there is a bijection

$$\left\{\varphi : \mathrm{Gal}\left(\overline{F}/F\right) \to S_3\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Isomorphism classes of}\\ \text{étale cubic algebras over } F\end{array}\right\}.$$

For any cubic algebra E let $S_E = Aut_F(E)$, which is a twisted form of S_3. An action of S_E on the algebraic group $Spin_8$ determines a simply-connected quasi-split form $H_E = Spin_8^E$ of the split group $Spin_8$ over F. We fix a Chevalley-Steinberg system of épinglage [3, Sections 4.1.3-4.1.4]

$$\left\{T_E, B_E, x_\gamma : \mathbb{G}_a \to \left(H_E\right)_\gamma, \gamma \in \Phi_{D_4}\right\},$$

where $T_E \subset B_E$ is a maximal torus contained in a Borel subgroup (both defined over F) and Φ_{D_4} are the roots of $H_E \otimes \overline{F} \cong Spin_8\left(\overline{F}\right)$. For any γ in the reduced root system of H_E we denote by F_γ the field of definition of γ.

We now give a more detailed description of H_E for the different kinds of étale cubic algebras over F in terms of the action of $\mathrm{Gal}\left(\overline{F}/F\right)$ on $H_E\left(\overline{F}\right)$.

1. $\underline{E = F \times F \times F}$: In this case H_E is the split reductive simply-connected group of type D_4 over F. It corresponds to the trivial action of $\mathrm{Gal}\left(\overline{F}/F\right)$. In this case we denote $\Gamma_E = \{1\}$. Also, in this case

$$F_{\alpha_1} = F_{\alpha_2} = F_{\alpha_3} = F_{\alpha_4} = F.$$

2. $\underline{E = F \times K}$: This is the case where $E = F \times K$ with K a quadratic (and hence Galois) extension of F. It is enough to define an action of $\Gamma_E = \mathrm{Gal}\left(K/F\right) = \langle\sigma\rangle$ on $Spin_8\left(K\right)$. This action is determined by

$$\begin{aligned}
\sigma\left(x_{\alpha_1}\left(k\right)\right) &= x_{\alpha_1}\left(\sigma\left(k\right)\right)\\
\sigma\left(x_{\alpha_2}\left(k\right)\right) &= x_{\alpha_2}\left(\sigma\left(k\right)\right)\\
\sigma\left(x_{\alpha_3}\left(k\right)\right) &= x_{\alpha_4}\left(\sigma\left(k\right)\right)\\
\sigma\left(x_{\alpha_4}\left(k\right)\right) &= x_{\alpha_3}\left(\sigma\left(k\right)\right).
\end{aligned}$$

In this case

$$F_{\alpha_1} = F_{\alpha_2} = F, \quad F_{\alpha_3} = F_{\alpha_4} = K.$$

Here, in accordance with **(CT)**, we single out the root α_1 from α_3 and α_4. The fact that each choice of distinct root results in an isomorphic algebraic group follows from *triality*. In what follows we choose α_1 to be the distinct root.

3. $\underline{E\text{ is a cubic Galois field extension:}}$ It is enough to define an action of $\Gamma_E = \mathrm{Gal}\left(E/F\right) = \langle\sigma : \sigma^3 = 1\rangle$ on $Spin_8\left(E\right)$. This action is determined by

$$\begin{aligned}
\sigma\left(x_{\alpha_2}\left(e\right)\right) &= x_{\alpha_2}\left(\sigma\left(e\right)\right)\\
\sigma\left(x_{\alpha_1}\left(e\right)\right) &= x_{\alpha_3}\left(\sigma\left(e\right)\right)\\
\sigma\left(x_{\alpha_3}\left(e\right)\right) &= x_{\alpha_4}\left(\sigma\left(e\right)\right)\\
\sigma\left(x_{\alpha_4}\left(e\right)\right) &= x_{\alpha_1}\left(\sigma\left(e\right)\right).
\end{aligned}$$

In this case

$$F_{\alpha_2} = F, \quad F_{\alpha_1} = F_{\alpha_3} = F_{\alpha_4} = E.$$

4. E is a cubic non-Galois field extension: In order to define $H_E(F)$ we first consider the Galois closure L of E over F as above with

$$\mathrm{Gal}(L/F) = \left\langle \sigma,\ \tau : \sigma^3 = 1,\ \tau^2 = 1,\ (\sigma\tau)^2 = 1 \right\rangle .$$

The action of $\Gamma_E = \mathrm{Gal}(L/F)$ on $Spin_8(L)$ is determined by

$$\begin{aligned} \sigma(x_{\alpha_2}(l)) &= x_{\alpha_2}(\sigma(l)), & \tau(x_{\alpha_2}(l)) &= x_{\alpha_2}(\tau(l)) \\ \sigma(x_{\alpha_1}(l)) &= x_{\alpha_3}(\sigma(l)), & \tau(x_{\alpha_1}(l)) &= x_{\alpha_3}(\tau(l)) \\ \sigma(x_{\alpha_3}(l)) &= x_{\alpha_4}(\sigma(l)), & \tau(x_{\alpha_3}(l)) &= x_{\alpha_1}(\tau(l)) \\ \sigma(x_{\alpha_4}(l)) &= x_{\alpha_1}(\sigma(l)), & \tau(x_{\alpha_4}(l)) &= x_{\alpha_4}(\tau(l)) . \end{aligned}$$

Here we singled out α_4 from α_1 and α_3 this is akin to distinguishing τ from $\sigma\tau\sigma^2$ and $\sigma^2\tau\sigma$. In this case

$$F_{\alpha_2} = F, \quad F_{\alpha_1} = E, \quad F_{\alpha_3} = E^{\sigma}, \quad F_{\alpha_4} = E^{\sigma^2}.$$

We denote by W_{H_E} the Weyl group of H_E with respect to T_E. It is generated by the simple reflections along the simple roots and we denote

$$w_{i_1,..,i_k} = w[i_1, ..., i_k] = w_{\gamma_{i_1}} \cdots w_{\gamma_{i_k}}.$$

We have a short exact sequence that splits

$$1 \to H_E \to \mathrm{Aut}(H_E) \to S_E \to 1.$$

Forming the semidirect product $H_E \rtimes S_E$ it holds that $G \cong \mathrm{Cent}_{H_E \rtimes S_E}(S_E)$. This gives a natural embedding

$$G \hookrightarrow H_E.$$

Moreover, (G, S_E) forms a dual reductive pair in $H_E \rtimes S_E$.

Also, it holds that $B_E \cap G = B$ and there exists an Heisenberg parabolic subgroup $P_E = M_E \cdot U_E$ of H_E so that $P_E \cap G = P$, $M_E \cap G = M$ and $U_E \cap G = U$. In particular

$$M_E \cong \left\{ g \in \mathrm{Res}_{E/F}\,\mathrm{GL}_2 : \det(g) \in \mathbb{G}_m \right\} .$$

1.8 The Degenerate Eisenstein Series

Fix a Hecke character $\chi : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$. We consider the normalized induction

$$I_{P_E}(\chi, s) = \mathrm{Ind}_{P_E(\mathbb{A})}^{H_E(\mathbb{A})}(\chi \circ \det_{M_E}) \left|\det_{M_E}\right|^{s+\frac{5}{2}},$$

where $\det_{M_E}$ is the determinant character associated with the Levi subgroup M_E. Note that for the modulus character of P_E it holds that $\delta_{P_E}|_{M_E} = \left|\det_{M_E}\right|^5$.

For any K-finite standard section $f_s \in I_{P_E}(\chi, s)$ we define the following degenerate Eisenstein series

$$\mathcal{E}_E(\chi, f_s, s, g) = \sum_{\gamma \in P_E(F)\backslash H_E(F)} f_s(\gamma g). \tag{1.8}$$

This series converges for $\mathfrak{Re}(s) \gg 0$ and admits a meromorphic continuation to the whole complex plane. For a finite set of places $S \subset \mathcal{P}$, containing all Archimedean places and all places ν where χ_ν is ramified, we normalize the Eisenstein series as follows

$$\mathcal{E}_E^*(\chi, f_s, s, g) = j_E^S(\chi, s)\, \mathcal{E}_E(\chi, f_s, s, g),$$

where

$$j_E^S(\chi, s) = \begin{cases} \mathcal{L}_F^S\left(s+\frac{5}{2}, \chi\right) \mathcal{L}_F^S\left(s+\frac{3}{2}, \chi\right)^2 \mathcal{L}_F^S\left(2s+1, \chi^2\right), & E = F \times F \times F \\ \mathcal{L}_F^S\left(s+\frac{5}{2}, \chi\right) \mathcal{L}_K^S\left(s+\frac{3}{2}, \chi \circ \mathrm{Nm}_{K/F}\right) \mathcal{L}_F^S\left(2s+1, \chi^2\right), & E = F \times K \\ \frac{\mathcal{L}_F^S\left(s+\frac{5}{2}, \chi\right) \mathcal{L}_E^S\left(s+\frac{3}{2}, \chi \circ \mathrm{Nm}_{E/F}\right) \mathcal{L}_F^S\left(2s+1, \chi^2\right)}{\mathcal{L}_F^S\left(s+\frac{3}{2}, \chi\right)}, & E \text{ field} \end{cases}.$$

For any standard section $f_s \in I_{P_E}(\chi, s)$ we write $f_s^* = j_E^S(\chi, s) f_s$ for the normalized section.

Remark 1.8.1. Let L be a quadratic or cubic extension of F. Assume that for some $\nu \in \mathcal{P}$ it holds that:

- L_ν is a direct product of unramified Galois extension of F_ν.
- χ_ν is unramified.

Note that $\left(\chi \circ \mathrm{Nm}_{L/F}\right)_\nu = \chi_\nu \circ \mathrm{Nm}_{L_\nu/F_\nu}$. It is then easy to check that the following holds:

- If ν splits in L we have

$$\mathcal{L}_{L_\nu}\left(s,\chi_\nu\circ\mathrm{Nm}_{L_\nu/F_\nu}\right)=\mathcal{L}_{F_\nu}\left(s,\chi_\nu\right)^{[L:F]}.$$

- If ν is inert in L we have

$$\mathcal{L}_{L_\nu}\left(s,\chi_\nu\circ\mathrm{Nm}_{L_\nu/F_\nu}\right)=\mathcal{L}_{F_\nu}\left([L:F]\cdot s,\chi_\nu^{[L:F]}\right).$$

- If L is a non-Galois extension of F, a quadratic extension K is associated with it as explained in Section 1.3. If $L_\nu=F_\nu\times K_\nu$ we have

$$\mathcal{L}_{L_\nu}\left(s,\chi_\nu\circ\mathrm{Nm}_{L_\nu/F_\nu}\right)=\mathcal{L}_{F_\nu}\left(s,\chi_\nu\right)\mathcal{L}_{F_\nu}\left([K:F]\cdot s,\chi_\nu^{[K:F]}\right)=\mathcal{L}_{F_\nu}\left(s,\chi_\nu\right)\mathcal{L}_{F_\nu}\left(2s,\chi_\nu^2\right).$$

Remark 1.8.2. When a Rankin-Selberg integral $\mathcal{Z}$ involves the Eisenstein series $\mathcal{E}\left(f_s,g,s\right)$, the unramified computation for the spherical section f^0 normalized by $f^0\left(1\right)=1$ often returns

$$\mathcal{Z}_\nu\left(\phi^0,f^0,s\right)=\frac{\mathcal{L}\left(s,\pi_\nu,\rho\right)}{j\left(s\right)},$$

where the factor $j\left(s\right)$ does not depend on π_v. It is customary to define the normalized Eisenstein series by $\mathcal{E}^*\left(f_s,g,s\right)=j\left(s\right)\mathcal{E}\left(f_s,g,s\right)$.

Recall that, due to the Gindikin-Karpelevich formula Corollary 6.1.7, the local intertwining operator $M_{w_0}\left(s\right)$ acts on a spherical vector as multiplication by a rational function $c\left(w_0,q^{-s}\right)$.

It was conjectured in [21] that the factor $j(s)$ is the numerator of the rational function $c\left(w_0,q^{-s}\right)$. In our case, this conjecture holds for all ireducible cuspidal representations π as can be seen from Theorem 2.0.2. At first glance, in the case that E is a field the conjecture seems to be violated. A closer inspection would reveal that, in fact, the conjecture holds in this case too as explained in the following discussion. The Gindikin-Karpelevich factor in this case being

$$\frac{\mathcal{L}_F^S\left(s-\frac{3}{2},\chi\right)\mathcal{L}_F^S\left(s+\frac{3}{2},\chi\right)\mathcal{L}_E^S\left(s-\frac{1}{2},\chi\circ\mathrm{Nm}_{E/F}\right)\mathcal{L}_F^S\left(2s,\chi^2\right)}{\mathcal{L}_F^S\left(s-\frac{1}{2},\chi\right)\mathcal{L}_F^S\left(s+\frac{5}{2},\chi\right)\mathcal{L}_E^S\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}_{E/F}\right)\mathcal{L}_F^S\left(2s+1,\chi^2\right)}.$$

We note that this factor can be rewritten as

$$\frac{\left(\frac{\mathcal{L}_F^S\left(s-\frac{3}{2},\chi\right)\mathcal{L}_E^S\left(s-\frac{1}{2},\chi\circ\mathrm{Nm}_{E/F}\right)\mathcal{L}_F^S\left(2s,\chi^2\right)}{\mathcal{L}_F^S\left(s-\frac{1}{2},\chi\right)}\right)}{\left(\frac{\mathcal{L}_F^S\left(s+\frac{5}{2},\chi\right)\mathcal{L}_E^S\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}_{E/F}\right)\mathcal{L}_F^S\left(2s+1,\chi^2\right)}{\mathcal{L}_F^S\left(s+\frac{3}{2},\chi\right)}\right)}$$

and in this form the denominator is indeed $j_E^S(\chi, s)$. We further note that if E is a Galois field extension then

$$\frac{\mathcal{L}_E^S\left(s, \chi \circ \mathrm{Nm}_{E/F}\right)}{\mathcal{L}_F^S(s,\chi)} = \mathcal{L}_F^S(s, \chi_E\chi)\, \mathcal{L}_F^S\left(s, \chi_E^2\chi\right)$$

and hence the Gindikin-Karpelevich can be rewritten as follows

$$\frac{\mathcal{L}_F^S\left(s-\frac{3}{2},\chi\right)\mathcal{L}_F^S\left(s-\frac{1}{2},\chi_E\chi\right)\mathcal{L}_F^S\left(s-\frac{1}{2},\chi_E^2\chi\right)\mathcal{L}_F^S\left(2s,\chi^2\right)}{\mathcal{L}_F^S\left(s+\frac{5}{2},\chi\right)\mathcal{L}_F^S\left(s+\frac{3}{2},\chi_E^2\chi\right)\mathcal{L}_F^S\left(s+\frac{3}{2},\chi_E\chi\right)\mathcal{L}_F^S\left(2s+1,\chi^2\right)}$$

and

$$j_E^S(\chi, s) = \mathcal{L}_F^S\left(s+\frac{5}{2},\chi\right)\mathcal{L}_F^S\left(s+\frac{3}{2},\chi_E^2\chi\right)\mathcal{L}_F^S\left(s+\frac{3}{2},\chi_E\chi\right)\mathcal{L}_F^S\left(2s+1,\chi^2\right).$$

If E/F is a non-Galois field extension then a similar argument can be made. Let σ denote the automorphic representation of $GL_2(\mathbb{A})$ corresponding to the 2-dimensional irreducible representation of $Gal(L/F) \equiv S_3$ according to [31, Theorem 3]. It then holds that

$$\frac{\mathcal{L}_E^S\left(s, \chi \circ \mathrm{Nm}_{E/F}\right)}{\mathcal{L}_F^S(s,\chi)} = \mathcal{L}^S\left(s, (\chi \circ \det)\,\sigma, id_{GL_2(\mathbb{C})}\right),$$

where $id_{GL_2(\mathbb{C})}$ is the identity map on $GL_2(\mathbb{C})$. It follows that the Gindikin-Karpelevich term in this case can be written as

$$\frac{\mathcal{L}_F^S\left(s-\frac{3}{2},\chi\right)\mathcal{L}^S\left(s-\frac{1}{2},(\chi\circ\det)\,\sigma, id_{GL_2(\mathbb{C})}\right)\mathcal{L}_F^S\left(2s,\chi^2\right)}{\mathcal{L}_F^S\left(s+\frac{5}{2},\chi\right)\mathcal{L}^S\left(s+\frac{3}{2},(\chi\circ\det)\,\sigma, id_{GL_2(\mathbb{C})}\right)\mathcal{L}_F^S\left(2s+1,\chi^2\right)}.$$

Finally we note that the denominator here equals $j_E(\chi, s)$.

Chapter 2

The Zeta Integral

Let π be an irreducible cuspidal representation. By this we mean an irreducible representation of $G(\mathbb{A})$ with a realization V_π in the space of cusp forms on $G(\mathbb{A})$. By [11, Theorem 2], for any ν
$in\mathcal{P}$ there exists an irreducible representation (π_ν, V_{π_ν}) of $G(F_\nu)$, almost all of the are unramified with respect to $G(\mathcal{O}_\nu)$, and an isomorphism of representations of $G(\mathbb{A})$:

$$i : \underset{\nu\in\mathcal{P}}{\otimes} V_{\pi_\nu} \xrightarrow{\sim} V_\pi.$$

We further assume that π supports the (U, Ψ_E)-Fourier coefficient for an étale cubic algebra E over F. We retain the assumption that the character $\Psi_E = \underset{\nu\in\mathcal{P}}{\otimes} \Psi_{E,\nu}$ of $U(\mathbb{A})$ corresponds to a triple (a, b, c) satisfying **(CT)**. Let $\chi = \underset{\nu\in\mathcal{P}}{\otimes} \chi_\nu$ be a Hecke character. For $\varphi \in \pi$ and a standard section $f_s \in I_{P_E}(\chi, s)$ we consider the integral

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \int\limits_{G(F)\backslash G(\mathbb{A})} \varphi(g)\, \mathcal{E}^*_E(\chi, s, f, g)\, dg. \tag{2.1}$$

Since φ is cuspidal, and hence rapidly decreasing, the meromorphic continuation of $\mathcal{Z}_E(\chi, s, \varphi, f)$ follows from that of $\mathcal{E}^*_E(\chi, s, f, g)$.

Let $\varphi = i\left(\underset{\nu\in\mathcal{P}}{\otimes} \varphi_\nu\right) \in \pi$, where $\varphi_\nu \in V_{\pi_\nu}$ and $f_s = \underset{\nu\in\mathcal{P}}{\otimes} f_{s,\nu} \in I_{P_E}(\chi, s)$ be pure tensors. Let $S \subset \mathcal{P}$ be a finite set of places such that for $\nu \notin S$ it holds that

- $2, 3 \not| \nu$ and $\nu \neq \infty$.

- E_ν is unramified over F_ν.
- G and H_E are defined over $\mathcal{O}_\nu$ and satisfy $G(\mathcal{O}_\nu) = H_E(\mathcal{O}_\nu) \cap G(F_\nu)$.
- π_ν and χ_ν are unramified and φ_ν and f_ν are spherical.
- ψ_ν is of conductor $\mathcal{O}_\nu$.
- Either $a = 0$ or $a \in \mathcal{O}_\nu^\times$ and similarly for b and c.
- Either $D_{a,b,c} = 0$ or $D_{a,b,c} \in \mathcal{O}_\nu^\times$.

Remark 2.0.1. It follows that either $N_{a,b,c} = 0$ or $N_{a,b,c} \in \mathcal{O}_\nu^\times$ for $\nu \notin S$. Also, note that if $N_{(a,b,c)} = 0$ then necessarily $D_{(a,b,c)} \neq 0$.

The main result of this thesis is

Theorem 2.0.2. *Let π, φ, f and $S \subset \mathcal{P}$ be as above. Then*

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \mathcal{L}^S\left(s + \frac{1}{2}, \pi, \chi, \mathfrak{st}\right) d_S(\chi, s, \Psi_E, \varphi, f), \tag{2.2}$$

where $d_S(\chi, s, \Psi_E, \varphi, f)$ is some meromorphic function which depends only on the data $\varphi_S = \otimes_{\nu\in S}\varphi_\nu$, $f_S = \otimes_{\nu\in S} f_{s,\nu}$ at ramified places. Moreover, for any s_0 there exist vectors φ_S, f_S such that $d_S(\chi, s, \Psi_E, \varphi, f)$ is analytic in a neighborhood of s_0 and $d_S(\chi, s_0, \Psi_E, \varphi_S, f_S) \neq 0$. In particular, $\mathcal{L}^S(s, \pi, \chi, \mathfrak{st})$ admits a meromorphic continuation to the whole complex plane.

Most of this thesis is devoted to the proof of this theorem. In this chapter we will outline the main ideas and defer the technical part to Chapters 3 through 5 and Appendix A. In this thesis we generalize the approach presented in [24] while dealing with some delicate issues arising in the non-split case. As noted in the introduction, this is done uniformly to the extent possible. The only place where it is necessary to consider the different étale cubic algebras separately is in Section 4.2 and Section 4.4 where the arithmetic difference plays an important role in the computation.

In Chapter 3 we prove the following theorem.

Theorem 2.0.3 (Unfolding). *For $\mathfrak{Re}(s) \gg 0$ it holds that*

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \int\limits_{U(\mathbb{A})\backslash G(\mathbb{A})} L_{\Psi_E}(\varphi)(g) F^*(\Psi_E, \chi, g, s)\, dg, \tag{2.3}$$

where

$$F^*(\Psi_E, \chi, g, s) = \int\limits_{\mathbb{A}} f_s^*(\mu_E x_{3\alpha+\beta}(r) g) \psi(r)\, dr \tag{2.4}$$

and $\mu_E = w_2 w_1 w_3 w_4 x_{\alpha_1}(a) x_{\alpha_3}(b) x_{\alpha_4}(c)$.

Remark 2.0.4. We note that $\mu_E \in H_E(F)$ due to **(CT)**.

In order to prove this, we describe the $G(F)$-orbits in $P_E(F)\backslash H_E(F)$. This is done in Section 3.1.

Remark 2.0.5. Note that if the section f is factorizable then so is F^*. In particular

$$F^*(\Psi_E, \chi, g, s) = \prod_{\nu \in \mathcal{P}} F_\nu^*(\Psi_{E,\nu}, \chi_\nu, g_\nu, s),$$

where

$$F_\nu^*(\Psi_{E,\nu}, \chi_\nu, g_\nu, s) = \int\limits_{F_\nu} f_{s,\nu}^*(\mu_E x_{3\alpha+\beta}(r) g_\nu) \psi_\nu(r)\, dr.$$

On the other hand, the integral in Equation (2.3) is *a priori* not factorizable since the space $\mathrm{Hom}_{U(\mathbb{A})}(\pi, \mathbb{C}_\Psi)$ is not necessarily one-dimensional and may even be infinite dimensional. Nevertheless, we will show that the integral is factorizable. This follows from an inductive process suggested in [35] and from the following two results.

Theorem 2.0.6 (Unramified Computation). *Let π_ν be an irreducible unramified representation of $G(F_\nu)$, and let v_0 be a fixed spherical vector in π_ν. There exists $s_0 \in \mathbb{R}$ such that for any $\mathfrak{Re}(s) > s_0$ and* **any** *$\Lambda \in \mathrm{Hom}_{U(F_\nu)}\left(\pi_\nu, \mathbb{C}_{\Psi_{E,\nu}}\right)$ it holds that*

$$\int\limits_{U(F_\nu)\backslash G(F_\nu)} F_\nu^*(\Psi_{E,\nu}, \chi_\nu, g, s) \Lambda(\pi_\nu(g) v_0)\, dg = \mathcal{L}\left(s + \frac{1}{2}, \pi_\nu, \chi_\nu, \mathfrak{st}\right) \Lambda(v_0). \tag{2.5}$$

Remark 2.0.7. The case of a split Fourier coefficient and $\chi \equiv \mathbb{1}$ is dealt with in [24]. The function F^* has a different form there due to the different choice of representative for the open orbit that gives rise to a character Ψ_s in the $M(F)$-orbit of $\Psi_{F\times F\times F}$. In Chapter 4 we relate $F_\nu^*(\Psi_{E,\nu}, \mathbb{1}, g, s)$ to $F_\nu^*(g, s)$ defined in [24].

Theorem 2.0.8 (Ramified Computation)**.** *For any $s_0 \in \mathbb{C}$ there exist data φ_S and f_S such that $d_S(\chi, s, \Psi_E, \varphi, f)$ is entire and non-vanishing in a neighborhood of s_0.*

Theorem 2.0.6 is proved in Chapter 4 in conjunction with the appendices. Theorem 2.0.8 is proved in Chapter 5. We now show how to derive the main theorem from the other results presented in this chapter.

Proof of Theorem 2.0.2. By Theorem 2.0.3

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \lim_{\substack{\longrightarrow \\ S\subset\Omega\subset\mathcal{P} \\ |\Omega|<\infty}} \int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} L_{\Psi_E}(\varphi)(g)\, F^*_\Omega\left(\Psi_{(a,b,c),\Omega}, \chi_\Omega, g, s\right) dg\ , \tag{2.6}$$

where $G(\mathbb{A})_\Omega = \prod_{\nu\in\Omega} G(F_\nu)$ and

$$F^*_\Omega(\Psi_{E,\Omega}, \chi_\Omega, g, s) = j_{E,\Omega}(s) \int_{F_\Omega} f_s(\mu_E x_{3\alpha+\beta}(r)\, g)\, \psi_\Omega(r)\, dr,$$

where

$$j_{E,\Omega} = \prod_{\nu\in\Omega} j_{E,\nu}, \quad \psi_\Omega = \otimes_{\nu\in\Omega}\psi_\nu.$$

Fix $s_0 \in \mathbb{R}$ such that the right hand side of Equation (2.3) converges for $\mathfrak{Re}\, s > s_0$. The integrals on the right hand side of Equation (2.6) must also converge there. Fix a finite subset $S \subseteq \Omega \subset \mathcal{P}$ and $\nu \notin \Omega$. Also fix $s_1 \in \mathbb{R}$ such that Theorem 2.0.6 holds for $\mathfrak{Re}\, s > s_1$ and π_ν. It holds that

$$\begin{aligned}
&\int_{U(\mathbb{A})_{\Omega\cup\{\nu\}} \backslash G(\mathbb{A})_{\Omega\cup\{\nu\}}} L_{\Psi_E}(\varphi)(g)\, F^*_{\Omega\cup\{\nu\}}\left(\Psi_{E,\Omega\cup\{\nu\}}, \chi_{\Omega\cup\{\nu\}}, g, s\right) dg \\
&= \int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} \int_{U(F_\nu)\backslash G(F_\nu)} L_{\Psi_E}(\varphi)(gg_\nu)\, F^*_{\Omega\cup\{\nu\}}\left(\Psi_{E,\Omega\cup\{\nu\}}, \chi_{\Omega\cup\{\nu\}}, gg_\nu, s\right) dg_\nu\, dg \\
&= \int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} F^*_\Omega(\Psi_{E,\Omega}, \chi_\Omega, g, s) \int_{U(F_\nu)\backslash G(F_\nu)} L_{\Psi_E}(\varphi)(gg_\nu)\, F^*_\nu(\Psi_{E,\nu}, \chi_\nu, g_\nu, s)\, dg_\nu\, dg \\
&= \mathcal{L}\left(s+\frac{1}{2}, \pi_\nu, \mathfrak{st}\right) \int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} L_{\Psi_E}(\varphi)(g)\, F^*_\Omega(\Psi_{E,\Omega}, \chi_\Omega, g, s)\, dg\ ,
\end{aligned}$$

where the last equality is due to Theorem 2.0.6. A priori the last equality holds only for $\mathfrak{Re}\, s > \max\{s_0, s_1\}$, but since $\mathcal{L}\left(s+\frac{1}{2}, \pi_\nu, \mathfrak{st}\right)$ is a meromorphic function the equality actually holds for $\mathfrak{Re}\, s > s_0$. Plugging this into Equation (2.6) we get

$$\begin{aligned} \mathcal{Z}_E\left(\chi, s, \varphi, f\right) &= \lim_{\substack{\longrightarrow \\ S\subset\Omega\subset\mathcal{P} \\ |\Omega|<\infty}} \prod_{\nu\in\Omega\setminus S} \mathcal{L}\left(s+\frac{1}{2}, \pi_\nu, \chi, \mathfrak{st}\right) \int_{U(\mathbb{A})_S\setminus G(\mathbb{A})_S} L_{\Psi_E}\left(\varphi\right)\left(g\right) F_S^*\left(\Psi_{E,S}, \chi_S, g, s\right)\, dg \\ &= \mathcal{L}^S\left(s+\frac{1}{2}, \pi, \chi, \mathfrak{st}\right) \int_{U(\mathbb{A})_S\setminus G(\mathbb{A})_S} L_{\Psi_E}\left(\varphi\right)\left(g\right) F_S^*\left(\Psi_{E,S}, \chi_S, g, s\right)\, dg\ . \end{aligned}$$

We finish the proof by fixing our datum according to Theorem 2.0.8 and taking

$$d_S\left(\chi, s, \Psi_E, \varphi, f\right) = \int_{U(\mathbb{A})_S\setminus G(\mathbb{A})_S} L_{\Psi_E}\left(\varphi\right)\left(g\right) F_S^*\left(\Psi_{E,S}, \chi_S, g, s\right)\, dg\ .$$

□

Chapter 3

The G_2 Orbits in $P_E\backslash H_E$ and the Unfolding of the Zeta Integral

In this chapter we prove

Theorem 2.0.3 . *For* $\mathfrak{Re}(s) \gg 0$ *it holds that*

$$\mathcal{Z}_E(\chi, s, \varphi, f) = \int\limits_{U(\mathbb{A})\backslash G(\mathbb{A})} L_{\Psi_E}(\varphi)(g) F^*(\Psi_E, \chi, g, s)\, dg, \tag{3.1}$$

where

$$F^*(\Psi_E, \chi, g, s) = \int\limits_{\mathbb{A}} f_s^*(\mu_E x_{3\alpha+\beta}(r) g)\, \psi(r)\, dr \tag{3.2}$$

and

$$\mu_E = w_2 w_1 w_3 w_4 x_{\alpha_1}(a) x_{\alpha_3}(b) x_{\alpha_4}(c).$$

Before doing this, we first parametrize the points in the double coset space $P_E(F)\backslash H_E(F)/G(F)$.

3.1 G_2 Orbits in $P_E\backslash H_E$

In order to unfold of Equation (2.1) we need to describe $P_E(F)\backslash H_E(F)/G(F)$ for any étale cubic algebra E over F. In order to do this, we first give a description of $P_E(\overline{F})\backslash H_E(\overline{F})/G(\overline{F})$. Although $P_E(\overline{F})\backslash H_E(\overline{F})/G(\overline{F})$ is independent of E, the representatives we choose depend on E. The method of Galois descent will be used in order to compute $(P_E\backslash H_E)(F)/G(F)$.

Denote $P_E\backslash H_E$ by X_E; this is a projective variety with a right G-action. Note that $X_E(F) = (P_E\backslash H_E)(F) = P_E(F)\backslash H_E(F)$ due to [2, Theorem 4.13a]. Let $Q = L\cdot V$ be the non-Heisenberg maximal parabolic subgroup of G. Its Levi part $L \cong GL_2$ is generated by the root β. The unipotent radical V of Q is a three-step unipotent group, we denote its commutator $[V,V]$ by R. We recall from [27, Lemma 2.1] that $X_E\left(\overline{F}\right)$ has five $G\left(\overline{F}\right)$-orbits, given as follows.

Lemma 3.1.1. *The following is a list of representatives of the $G\left(\overline{F}\right)$-orbits in $X_E\left(\overline{F}\right)$ and their stabilizers:*

1. $\mu = 1$ *and the stabilizer of* $P_E\left(\overline{F}\right)\mu G\left(\overline{F}\right)$ *is* $G^\mu = P$.
2. $\mu = w_2w_1, w_2w_3, w_2w_4$ *and the stabilizer of* $P_E\left(\overline{F}\right)\mu G\left(\overline{F}\right)$ *is* $G^\mu = LR$.
3. $\mu = w_2w_3x_{-\alpha_1}(1)$ *is a representative of the open orbit and the stabilizer of* $P_E\left(\overline{F}\right)\mu G\left(\overline{F}\right)$ *is* $G^\mu = T_{3\alpha+2\beta}\cdot U^\mu$, *where*

$$T_{3\alpha+2\beta} = \left\{h_{3\alpha+2\beta}(t) : t\in \overline{F}^\times\right\},\quad U^\mu = \left\{u(r_1,r_2,-r_2,r_4,r_5) : r_i\in\overline{F}\right\}.$$

We now give a different set of representatives for these orbits. For a triple $(a,b,c)\in \overline{F}\times\overline{F}\times\overline{F} = \left(\mathrm{Res}_{E/F}\mathbb{G}_a\right)\left(\overline{F}\right)$ we denote

$$\mu(a,b,c) = w_2w_1w_3w_4x_{\alpha_1}(a)\,x_{\alpha_3}(b)\,x_{\alpha_4}(c).$$

For $x,x'\in X_E\left(\overline{F}\right)$ we write $x\sim x'$ if they lie in the same $G\left(\overline{F}\right)$-orbit.

Lemma 3.1.2. *Let* $(a,b,c)\in\overline{F}\times\overline{F}\times\overline{F}$ *then*

1. $a=b=c$ *if and only if* $\mu(a,b,c)\sim 1$.
2.
   - $a=b\neq c$ *if and only if* $\mu(a,b,c)\sim w_2w_4$.
   - $a\neq b=c$ *if and only if* $\mu(a,b,c)\sim w_2w_1$.
   - $a=c\neq b$ *if and only if* $\mu(a,b,c)\sim w_2w_3$.
3. a, b *and* c *are distinct if and only if* $\mu(a,b,c)\sim w_2w_3x_{-\alpha_1}(1)$.

Remark 3.1.3. The proof of this lemma relies on the use of Möbius transformations. As

$$M_E\left(\overline{F}\right) \cong \left(GL_2 \times GL_2 \times GL_2\right)^0\left(\overline{F}\right) = \left\{g \in \left(GL_2 \times GL_2 \times GL_2\right)\left(\overline{F}\right) : \det\left(g\right) \in \overline{F}^{\times}\right\},$$

we have a natural map

$$\begin{aligned}\left(GL_2 \times GL_2 \times GL_2\right)^0\left(\overline{F}\right) &\xrightarrow{\Upsilon} P_E\left(\overline{F}\right)\backslash H_E\left(\overline{F}\right)\\ m &\mapsto P_E\left(\overline{F}\right) w_2\, m.\end{aligned}$$

We also note that $w_2\left(M_E\left(\overline{F}\right) \cap B_E\left(\overline{F}\right)\right) w_2 \subset P_E\left(\overline{F}\right)$ and hence this map factors through

$$\left(B_0 \times B_0 \times B_0\right)^0\left(\overline{F}\right) \backslash \left(GL_2 \times GL_2 \times GL_2\right)^0\left(\overline{F}\right),$$

where B_0 denotes the Borel subgroup of GL_2. This quotient admits a diagonal action of GL_2 from the right. On the other hand, $M \cong GL_2$ acts from the right on $P_E\left(\overline{F}\right)\backslash H_E\left(\overline{F}\right)$. The map Υ is evidently GL_2-equivariant. Namely, for any $\left(m_1, m_2, m_3\right) \in \left(GL_2 \times GL_2 \times GL_2\right)^0\left(\overline{F}\right)$ and any $m \in GL_2\left(\overline{F}\right)$ it holds that

$$\Upsilon\left(\left(m_1, m_2, m_3\right) \cdot m\right) = \Upsilon\left(\left(m_1, m_2, m_3\right)\right) \cdot m.$$

Recall that due to the Bruhat decomposition, $\left(B_0\left(\overline{F}\right)\backslash GL_2\left(\overline{F}\right)\right) \cong \mathbb{P}^1\left(\overline{F}\right)$. This identification can be realized as

$$x \longleftrightarrow B_0\left(\overline{F}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \infty \longleftrightarrow B_0\left(\overline{F}\right).$$

Hence, the right action of $M\left(\overline{F}\right)$ on $P_E\left(\overline{F}\right)\backslash H_E\left(\overline{F}\right)$ is induced from the action of $GL_2\left(\overline{F}\right)$ on triples of points in $\mathbb{P}^1\left(\overline{F}\right)$. In particular, using these projective coordinates, we note that $w_2 w_1 = \Upsilon\left(0, \infty, \infty\right)$, $w_2 w_3 = \Upsilon\left(\infty, 0, \infty\right)$, $w_2 w_4 = \Upsilon\left(\infty, \infty, 0\right)$ and $w_2 w_3 x_{-\alpha_1}\left(1\right) = \Upsilon\left(1, 0, \infty\right)$. As for the trivial orbit, the identity 1 of $H_E\left(\overline{F}\right)$ is not in the image of Υ but $1 \sim \Upsilon\left(\infty, \infty, \infty\right)$.

Furthermore, the orbits of $GL_2\left(\overline{F}\right)$ in $\mathbb{P}^1\left(\overline{F}\right) \times \mathbb{P}^1\left(\overline{F}\right) \times \mathbb{P}^1\left(\overline{F}\right)$ correspond to the items in Lemma 3.1.2. Namely, $\left(\mathbb{P}^1\left(\overline{F}\right) \times \mathbb{P}^1\left(\overline{F}\right) \times \mathbb{P}^1\left(\overline{F}\right)\right)/GL_2\left(\overline{F}\right)$ is given by:

1. $\left\{(a,a,a) : a \in \mathbb{P}^1\left(\overline{F}\right)\right\}$.

2.
   - $\left\{(a,a,c) : a,c \in \mathbb{P}^1\left(\overline{F}\right) \text{ distinct}\right\}$.
   - $\left\{(a,b,b) : a,b \in \mathbb{P}^1\left(\overline{F}\right) \text{ distinct}\right\}$.
   - $\left\{(a,b,a) : a,b \in \mathbb{P}^1\left(\overline{F}\right) \text{ distinct}\right\}$.

3. $\left\{(a,b,c) : a,b,c \in \mathbb{P}^1\left(\overline{F}\right) \text{ distinct}\right\}$.

In fact, the lemma proves that there is a natural bijection

$$(B_0 \times B_0 \times B_0)^0\left(\overline{F}\right) \backslash (GL_2 \times GL_2 \times GL_2)^0\left(\overline{F}\right)/GL_2\left(\overline{F}\right) \longleftrightarrow P_E\left(\overline{F}\right)\backslash H_E\left(\overline{F}\right)/G\left(\overline{F}\right). \tag{3.3}$$

Proof. 1. In this case $\mu(a,a,a) = w_2w_1w_3w_4x_\alpha(a) \in G_2\left(\overline{F}\right)$ and hence

$$\mu(a,a,a) \sim 1.$$

2. We prove, for example, that $\mu(a,a,c) \sim w_2w_4$ for $a \neq c$. In this case, let

$$m(a,a,c) = x_\alpha(-a)\,w_\alpha h_\beta(a-c)\,x_\alpha(1) \in M\left(\overline{F}\right) \subset G\left(\overline{F}\right).$$

   One checks that

$$\mu(a,a,c)\,m(a,a,c)\,(w_2w_4)^{-1} \in B_E\left(\overline{F}\right)$$

   and hence

$$\mu(a,a,c) \sim w_2w_4.$$

3. Let

$$m_{a,b,c} = h_\alpha(b-a)\,h_\beta\left(\frac{(a-b)^3}{(a-c)(b-c)}\right)x_\alpha\left(c\frac{(a-b)}{(c-a)(b-c)}\right)w_\alpha x_\alpha\left(\frac{(a-c)}{(a-b)}\right) \in M\left(\overline{F}\right)$$

   One checks that

$$\mu(a,b,c)\,m_{a,b,c}\left(w_2w_3x_{-\alpha_1}(1)\right)^{-1} \in B_E\left(\overline{F}\right),$$

   hence

$$\mu(a,b,c) \sim w_2w_3x_{-\alpha_1}(1)\ .$$

Since Lemma 3.1.1 gives a list of representatives of orbits, the other direction for all items follows immediately. □

For the rest of this thesis we fix a triple (a, b, c) for every E as in **(CT)** and write $\mu_E = \mu(a, b, c)$.

Corollary 3.1.4. $\mathrm{Stab}_{G(\overline{F})}\left(P_E\left(\overline{F}\right)\mu_E\right) = T_{3\alpha+2\beta}\cdot\ker\Psi_E$.

Remark 3.1.5. Here, by abuse of notation, we denote by $\ker\Psi_E$ the kernel of the map

$$u(r_1, r_2, r_3, r_4, r_5) \mapsto r_4 - T_{(a,b,c)}r_3 + D_{(a,b,c)}r_2 - N_{(a,b,c)}r_1.$$

Proof. Recall from [24] that Ψ_s is given by

$$\Psi_s(u(r_1, r_2, r_3, r_4, r_5)) = \psi(r_2 + r_3).$$

Again, by abuse of notation, we denote by $\ker\Psi_s$ the kernel of the map

$$u(r_1, r_2, r_3, r_4, r_5) \mapsto r_2 + r_3.$$

Note that

$$\mathrm{Stab}_{G(\overline{F})}\left(P_E\left(\overline{F}\right)w_2w_3x_{-\alpha_1}(1)\right) = T_{3\alpha+2\beta}\cdot\ker\Psi_s.$$

We also note that $\ker\Psi_s$ and $\ker\Psi_E$ are two algebraic groups defined over F.

From Lemma 3.1.2

$$P_E\left(\overline{F}\right)w_2w_3x_{-\alpha_1}(1) = P_E\left(\overline{F}\right)\mu_E m_{a,b,c}$$

and hence

$$\mathrm{Stab}_{G(\overline{F})}\left(P_E\left(\overline{F}\right)\mu_E\right) = m_{a,b,c}\,\mathrm{Stab}_{G(\overline{F})}\left(P_E\left(\overline{F}\right)w_2w_3x_{-\alpha_1}(1)\right)m_{a,b,c}^{-1}.$$

On the other hand,

$$\Psi_E(u) = \Psi_s\left(m_{a,b,c}^{-1}um_{a,b,c}\right)$$

and hence

$$\ker\Psi_E = m_{a,b,c}^{-1}\left(\ker\Psi_s\right)m_{a,b,c}.$$

Also, $T_{3\alpha+2\beta}$ is the center $\mathcal{Z}(M)$ of M and hence $m_{a,b,c}^{-1}T_{3\alpha+2\beta}m_{a,b,c} = T_{3\alpha+2\beta}$. □

We now move to describing the $G(F)$ orbits in $X_E(F)$. Denote by X_μ the G-orbit of $P_E(F)\mu \in X_E(F)$. Given $\mu \in H_E(F) \subseteq H_E(\overline{F})$ we have a short exact sequence

$$\{1\} \to \mathrm{Stab}_{G(\overline{F})}(\mu) \to G_2(\overline{F}) \to X_\mu(\overline{F}) \to \{1\}.$$

Recall that the action of $\mathrm{Gal}(\overline{F}/F)$ on $H_E(\overline{F})$ and $X_E(\overline{F})$ factors through Γ_E. We then have ([1, Proposition II.4.7]) a long exact sequence in the cohomology

$$\{1\} \longrightarrow \mathrm{Stab}_{G(\overline{F})}(\mu)^{\Gamma_E} \longrightarrow G(F) \longrightarrow X_\mu(F) \longrightarrow \mathrm{H}^1\left(\Gamma_E, \mathrm{Stab}_{G(\overline{F})}\left(P_E(\overline{F})\mu\right)\right) \longrightarrow \mathrm{H}^1\left(\Gamma_E, G(\overline{F})\right) \longrightarrow \mathrm{H}^1\left(\Gamma_E, X_\mu(\overline{F})\right).$$

In particular, there is a bijection (of sets)

$$X_\mu(F)/G(F) \longleftrightarrow \ker\left[\mathrm{H}^1\left(\Gamma_E, \mathrm{Stab}_{G(\overline{F})}\left(P_E(\overline{F})\mu\right)\right) \to \mathrm{H}^1\left(\Gamma_E, X_\mu(\overline{F})\right)\right].$$

Since G is a split and reductive algebraic group, then according to [41, §8] it holds that $\mathrm{H}^1(\Gamma_E, G(F)) = \{1\}$ and hence $X_\mu(F)/G(F)$ is parametrized by $\mathrm{H}^1\left(\Gamma_E, \mathrm{Stab}_{G(F)}(\mu)\right)$.

Theorem 3.1.6. *The $G(F)$-orbits in $P_E(F)\backslash H_E(F)$ are*

1. *$\underline{E = F \times F \times F}$: $P_E(F)G(F)$, $P_E(F)w_2w_1G(F)$, $P_E(F)w_2w_3G(F)$, $P_E(F)w_2w_4G(F)$ and $P_E(F)w_2w_1w_3w_4x_{\alpha_1}(1)x_{\alpha_3}(-1)G(F)$.*

2. *$\underline{E = F \times K,\ K = F[\theta] \text{ a field}}$: $P_E(F)G(F)$, $P_E(F)w_2w_1G(F)$ and $P_E(F)w_2w_1w_3w_4x_{\alpha_3}(\theta)x_{\alpha_4}(\theta^\sigma)G(F)$.*

3. *$\underline{E = F[\theta] \text{ a field}}$: $P_E(F)G(F)$ and $P_E(F)w_2w_1w_3w_4x_{\alpha_1}(\theta)x_{\alpha_3}(\theta^\sigma)x_{\alpha_4}\left(\theta^{\sigma^2}\right)G(F)$.*

Proof.

1. This follows from Lemma 3.1.1 combined with Lemma 3.1.2.

2. We start by noting that $P_E(F)G(F)$, $P_E(F)w_2w_1G(F)$ and $P_E(F)w_2w_1w_3w_4x_{\alpha_3}(\theta)x_{\alpha_4}(\theta^\sigma)G(F)$ are the only $G(\overline{F})$-orbits in $P_E(\overline{F})\backslash H_E(\overline{F})$ that intersect $H_E(F)$. We need only show that for these μ-s, $P_E(F)\mu$ is one $G(F)$-orbit.

- $\underline{\mu = 1}$: In this case, $\mathrm{Stab}_G(\mu) = P$ and we have a short exact sequence

$$\{1\} \to U \to P \to M \to \{1\}.$$

$\mathrm{H}^1\left(\Gamma_E, U\left(\overline{F}\right)\right) = \{1\}$ since U is unipotent and $\mathrm{H}^1\left(\Gamma_E, M\left(\overline{F}\right)\right) = \{1\}$ due to Hilbert's Theorem 90 and thus $\mathrm{H}^1\left(\Gamma_E, P\left(\overline{F}\right)\right) = \{1\}$. Hence $X_1(F)/G(F) = \{1\}$.

- $\underline{\mu = w_2w_1}$: In this case, $\mathrm{Stab}_G(P_E\mu) = L\cdot R$ and we have a short exact sequence

$$\{1\} \to R \to \mathrm{Stab}_G(P_E\mu) \to L \to \{1\}.$$

$\mathrm{H}^1\left(\Gamma_E, R\left(\overline{F}\right)\right) = \{1\}$ since R is unipotent and $\mathrm{H}^1\left(\Gamma_E, L\left(\overline{F}\right)\right) = \{1\}$ due to Hilbert's Theorem 90 and thus $\mathrm{H}^1\left(\Gamma_E, \mathrm{Stab}_{G(\overline{F})}\left(P_E\left(\overline{F}\right)\mu\right)\right) = \{1\}$. Hence $X_\mu(F)/G(F) = \{P_E(F)\mu\}$.

- $\underline{\mu = P_E(F)\,w_2w_1w_3w_4x_{\alpha_3}(\theta)\,x_{\alpha_4}(\theta^\sigma)}$: In this case,

$$\mathrm{Stab}_G(P_E\mu) = T_{3\alpha+2\beta}\cdot\ker\Psi_E$$

and we have a short exact sequence

$$\{1\} \to \ker\Psi_E \to \mathrm{Stab}_G(P_E\mu) \to T_{3\alpha+2\beta} \to \{1\}.$$

$\mathrm{H}^1\left(\Gamma_E, (\ker\Psi_E)\left(\overline{F}\right)\right) = \{1\}$ since $\ker\Psi_E$ is unipotent and $\mathrm{H}^1\left(\Gamma_E, T_{3\alpha+2\beta}\left(\overline{F}\right)\right) = \{1\}$ due to Hilbert's Theorem 90. Thus $\mathrm{H}^1\left(\Gamma_E, \mathrm{Stab}_{G(\overline{F})}\left(P_E\left(\overline{F}\right)\mu\right)\right) = \{1\}$. Hence $X_\mu(F)/G(F) = \{P_E(F)\mu\}$.

3. This case is proven similarly.

□

The stabilizers of the representatives of orbits listed in the previous theorem are given below.

Remark 3.1.7. We recall that for $\mu \in H_E(F)$

$$\mathrm{Stab}_{G(F)}(P_E(F)\mu) = \mathrm{Stab}_{G(\overline{F})}\left(P_E\left(\overline{F}\right)\mu\right) \cap G(F).$$

Corollary 3.1.8. *1. It holds that* $\mathrm{Stab}_{G(F)}(P_E(F)) = P(F)$.

2. *It holds that*

$$\mathrm{Stab}_{G(F)}\left(P_E\left(F\right)w_2w_1\right) = \mathrm{Stab}_{G(F)}\left(P_E\left(F\right)w_2w_3\right) = \mathrm{Stab}_{G(F)}\left(P_E\left(F\right)w_2w_4\right) = L{\cdot}R,$$

if applicable.

3. *It holds that* $\mathrm{Stab}_{G(F)}\left(P_E\left(F\right)\mu_E\right) = T_{3\alpha+2\beta}\left(F\right)\cdot\ker\Psi_E\left(F\right)$.

3.2 The Unfolding of the Zeta Integral

In this section we prove Theorem 2.0.3. We denote by N_β the unipotent radical of the Borel subgroup $B\cap L$ of L.

Proof of Theorem 2.0.3. For $\mathfrak{Re}\left(s\right)\gg 0$ it holds that

$$\begin{aligned}\frac{1}{j_E\left(\chi,s\right)}\mathcal{Z}_E\left(\chi,s,\varphi,f\right) &= \int\limits_{G(F)\backslash G(\mathbb{A})}\varphi\left(g\right)\mathcal{E}_E\left(\chi,s,f,g\right)dg\\ &= \int\limits_{G(F)\backslash G(\mathbb{A})}\varphi\left(g\right)\sum_{\gamma\in P_E(F)\backslash H_E(F)}f_s\left(\gamma g\right)dg\\ &= \sum_{\mu\in P_E(F)\backslash H_E(F)/G(F)} I_\mu\left(\varphi,f_s\right),\end{aligned}$$

where

$$I_\mu\left(\varphi,f_s\right) = \int\limits_{G^\mu(F)\backslash G(\mathbb{A})}\varphi\left(g\right)f_s\left(\mu g\right)dg.$$

We now show that $I_\mu\left(\varphi,f_s\right)=0$ unless μ is a representative of the open orbit.

1. $\underline{\mu=1}$ Let $\mu=1$. Then

$$\begin{aligned}I_\mu\left(\varphi,f_s\right) &= \int\limits_{P(F)\backslash G(\mathbb{A})}\varphi\left(g\right)dg\\ &= \int\limits_{M(F)U(\mathbb{A})\backslash G(F)} f_s\left(g\right)\left(\int\limits_{U(F)\backslash U(\mathbb{A})}\varphi\left(ug\right)du\right)dg = 0,\end{aligned}$$

since φ is cuspidal.

2. $\underline{\mu \in \{w_2w_1, w_2w_3, w_2w_4\}}$: Let $\mu \in \{w_2w_1, w_2w_3, w_2w_4\}$ (when applicable). Then

$$I_\mu(\varphi, f_s) = \int\limits_{L(F)\cdot R(F)\backslash G(\mathbb{A})} \varphi(g) f_s(\mu g)\, dg$$
$$= \int\limits_{L(F)\cdot R(\mathbb{A})\backslash G(\mathbb{A})} f_s(\mu g) \left(\int\limits_{R(F)\backslash R(\mathbb{A})} \varphi(rg)\, dr \right) dg.$$

Recall from [39, Theorem 5] that

$$\int\limits_{R(F)\backslash R(\mathbb{A})} \varphi(rg)\, dr = \sum_{\nu \in N_\beta(F)\backslash L(F)} W_\psi(\varphi)(\nu g),$$

where $W_\psi(\varphi)$ is the standard Whittaker coefficient of φ. We then have

$$I_\mu(\varphi, f_s) = \int\limits_{L(F)\cdot R(\mathbb{A})\backslash G(\mathbb{A})} f_s(\mu g) \left(\sum_{\nu \in N_\beta(F)\backslash L(F)} W_\psi(\varphi)(\nu g) \right) dg$$
$$= \int\limits_{N_\beta(\mathbb{A})\cdot R(\mathbb{A})\backslash G(\mathbb{A})} f_s(\mu g) W_\psi(\varphi)(\nu g) \left(\int\limits_{N_\beta(F)\backslash N_\beta(\mathbb{A})} \psi(n)\, dn \right) dg = 0.$$

3. $\underline{\mu = \mu_E}$ Fix $\mu = \mu_E$ to be the representative of the open orbit as in Theorem 3.1.6. Also let $T^\mu = T_{3\alpha+2\beta}$ and $U^\mu = \ker \Psi_E$ so that $\mathrm{Stab}_G(P_E\mu) = T^\mu U^\mu$. It holds that

$$I_\mu(\varphi, f_s) = \int\limits_{T^\mu(F)U^\mu(\mathbb{A})\backslash G(\mathbb{A})} f_s(\mu g) \left(\int\limits_{U^\mu(F)\backslash U^\mu(\mathbb{A})} \varphi(ug)\, du \right) dg. \qquad (3.4)$$

We now expand the function given by the inner integral along the 1-dimensional subgroup generated by the root $3\alpha+\beta$. Note that $U = U^\mu \cdot U_{3\alpha+\beta}$ and Ψ_E is non-trivial on $U_{3\alpha+\beta}$ due to the assumption **(CT)**. In particular

$$\Psi_E(x_{3\alpha+\beta}(r)) = \psi(r).$$

Writing the Fourier expansion along this root yields

$$\varphi(g) = \sum_{a \in F} \int\limits_{F\backslash\mathbb{A}} \varphi(x_{3\alpha+\beta}(r) g) \overline{\psi(ar)} dr \quad \forall g \in G(\mathbb{A}).$$

Applying cuspidality of φ implies

$$\begin{aligned}\int\limits_{U^\mu(F)\backslash U^\mu(\mathbb{A})} \varphi(ug)\,du &= \int\limits_{U^\mu(F)\backslash U^\mu(\mathbb{A})} \sum_{a\in F} \int\limits_{F\backslash\mathbb{A}} \varphi(x_{3\alpha+\beta}(r)\,ug)\,\overline{\psi(ar)}dr\ du \\ &= \sum_{a\in F} \int\limits_{U^\mu(F)\backslash U^\mu(\mathbb{A})} \int\limits_{F\backslash\mathbb{A}} \varphi(x_{3\alpha+\beta}(r)\,ug)\,\overline{\psi(ar)}dr\ du \\ &= \sum_{a\in F^\times} \int\limits_{U^\mu(F)\backslash U^\mu(\mathbb{A})} \int\limits_{F\backslash\mathbb{A}} \varphi(h_{3\alpha+2\beta}(a)\,x_{3\alpha+\beta}(r)\,ug)\,\overline{\psi(ar)}dr\ du \\ &+ \int\limits_{U^\mu(F)\backslash U^\mu(\mathbb{A})} \int\limits_{F\backslash\mathbb{A}} \varphi(x_{3\alpha+\beta}(r)\,ug)\,dr\ du\end{aligned}$$

The second summand

$$\int\limits_{U^\mu(F)\backslash U^\mu(\mathbb{A})} \int\limits_{F\backslash\mathbb{A}} \varphi(x_{3\alpha+\beta}(r)\,ug)\,dr\ du = \int\limits_{U(F)\backslash U(\mathbb{A})} \varphi(ug)\,du = 0$$

vanish due to cuspidality. It follows that

$$\begin{aligned}\int\limits_{U^\mu(F)\backslash U^\mu(\mathbb{A})} \varphi(ug)\,du &= \sum_{a\in F^\times} \int\limits_{U^\mu(F)\backslash U^\mu(\mathbb{A})} \int\limits_{F\backslash\mathbb{A}} \varphi(x_{3\alpha+\beta}(ar)\,uh_{3\alpha+2\beta}(a)\,g)\,\overline{\psi(ar)}dr\ du \\ &= \sum_{a\in F^\times} \int\limits_{U(F)\backslash U(\mathbb{A})} \varphi(uh_{3\alpha+2\beta}(a)\,g)\,\overline{\Psi_E(u)}du\end{aligned}$$

Plugging this into Equation (3.4) yields

$$I_\mu(\varphi, f_s) = \int\limits_{T^\mu(F)U^\mu(\mathbb{A})\backslash G(\mathbb{A})} f_s(\mu g) \left(\sum_{a\in F^\times} \int\limits_{U(F)\backslash U(\mathbb{A})} \varphi(uh_{3\alpha+2\beta}(a)\,g)\,\overline{\Psi_E(u)}du \right) dg$$

Collapsing the sum with the outer integration, we conclude that the above equals

$$I_\mu(\varphi, f_s) = \int\limits_{U^\mu(\mathbb{A})\backslash G(\mathbb{A})} f_s(\mu g) \left(\int\limits_{U(F)\backslash U(\mathbb{A})} \varphi(ug)\,\overline{\Psi_E(u)}du \right) dg.$$

Since $U = U^{\mu} \cdot U_{3\alpha+\beta}$ we have

$$I_{\mu}(\varphi, f_s) = \int\limits_{U(\mathbb{A})\backslash G(\mathbb{A})} \left(\int\limits_{U(F)\backslash U(\mathbb{A})} \varphi(ug)\, \overline{\Psi_E(u)} du \right) \left(\int\limits_{\mathbb{A}} f_s(\mu x_{3\alpha+\beta}(r)\, g)\, \psi(r)\, dr \right) dg.$$

$\square$

Chapter 4

The Unramified Computation

In this chapter we prove

Theorem 2.0.6 . *Let $\nu \notin S$, and let v_0 be a fixed spherical vector in π_ν. There exists $s_0 \in \mathbb{R}$ such that for any $\mathfrak{Re}(s) > s_0$ and* **any** $\Lambda \in \mathrm{Hom}_{U(F_\nu)}\left(\pi_\nu, \mathbb{C}_{\Psi_{E,\nu}}\right)$ *it holds that*

$$\int\limits_{U(F_\nu)\backslash G(F_\nu)} F_\nu^*\left(\Psi_{E,\nu}, \chi_\nu, g, s\right) \Lambda\left(\pi_\nu\left(g\right) v_0\right) dg = \mathcal{L}\left(s + \frac{1}{2}, \pi_\nu, \chi_\nu, \mathfrak{st}\right) \Lambda\left(v_0\right). \tag{4.1}$$

We remind, from Chapter 2l, the reader that the assumption $\nu \notin S$ implies

- $2, 3 \not| \nu$ and $\nu \neq \infty$.
- E_ν is unramified over F_ν.
- G and H_E are defined over $\mathcal{O}_\nu$ and satisfy $G\left(\mathcal{O}_\nu\right) = H_E\left(\mathcal{O}_\nu\right) \cap G\left(F_\nu\right)$.
- π_ν and χ_ν are unramified and φ_ν and f_ν are spherical.
- ψ_ν is of conductor $\mathcal{O}_\nu$.
- Either $a = 0$ or $a \in \mathcal{O}_\nu^\times$ and similarly for b and c.
- Either $D_{a,b,c} = 0$ or $D_{a,b,c} \in \mathcal{O}_\nu^\times$.

For this chapter and the Appendix A we fix a place $\nu \notin S$. drop ν from all of the notations, i.e. we assume F to be a local non-Archimedean field, E an unramified Galois étale cubic algebra over F, π an irreducible unramified representation of $G\left(F\right)$, etc. We fix on $G\left(F\right)$ the unique Haar measure λ such that $\lambda\left(K\right) = 1$, where $K = G\left(\mathcal{O}\right)$.

4.1 The Generating Function and Its Approximation

In this section we describe the unramified computation and recall some results that where proven in my M.Sc. thesis and published in [24]. Most of the calculations needed for the proof of Theorem 2.0.6 can be found in this chapter; the rest are detailed in Appendix A.

Recall the Satake isomorphism between the spherical Hecke algebra $\mathcal{H} = \mathcal{H}(G, K)$ and the Grothendieck ring $Rep\left({}^{L}G\right)$, described in [23]. Denote by $A_j \in \mathcal{H}$ the elements corresponding to $Sym^j(\mathfrak{st})$ under the Satake isomorphism. In particular, for any unramified representation π and a spherical vector $v_0 \in \pi$ it holds that

$$\int\limits_{G(F)} A_j(g)\,\pi(g)\,v_0\,dg = \mathrm{Tr}\left(Sym^j(\mathfrak{st})(t_\pi)\right)v_0, \tag{4.2}$$

where $t_\pi \in {}^{L}G(\mathbb{C})$ is the Satake parameter of π.

For any such π, the Satake isomorphism induces an algebra homomorphism

$$\begin{aligned}\mathcal{H} &\to \mathbb{C}\\ f &\to \widehat{f}(\pi),\end{aligned}$$

where $\widehat{f}(\pi)$ is given by

$$\int\limits_{G(F)} f(g)\,\pi(g)\,v_0\,dg = \widehat{f}(\pi)\,v_0.$$

In particular, for any $f_1, f_2 \in \mathcal{H}$ it holds that $\widehat{f_1 * f_2} = \widehat{f_1}\cdot\widehat{f_2}$. The homomorphism $f \to \widehat{f}(\pi)$ can be extended linearly to a map of formal power series $\mathcal{H}[[T]] \to \mathbb{C}[[T]]$. Let $T = q^{-s}$.

Theorem 4.1.1. *For any finite order character χ of F, there exists a unique generating function $\Delta_{\chi,s} \in \mathcal{H}[[q^{-s}]]$, uniformly converging on a right half-plane, such that for any unramified representation π with a spherical vector v_0 and* ***any*** *functional Λ on π, it holds that*

$$\int\limits_{G(F)} \Delta_{\chi,s}(g)\,\pi(g)\,v_0 dg = \mathcal{L}(s,\pi,\chi,\mathfrak{st})\,v_0$$

for $\mathfrak{Re}(s) \gg 0$*, and in particular:*

$$\int_{G(F)} \Delta_{\chi,s}(g)\,\Lambda(\pi(g)\,v_0)\,dg = \mathcal{L}(s,\pi,\chi,\mathfrak{st})\,\Lambda(v_0)\,. \tag{4.3}$$

Proof. We have Poincaré's identity

$$\begin{aligned}\mathcal{L}(s,\pi,\chi,\mathfrak{st}) &= \frac{1}{\det\left(\mathbb{1} - q^{-s}\,\mathfrak{st}\left(t_{\pi\boxtimes\chi}\right)\right)} = \prod_{i=1}^{7} \frac{1}{1 - q^{-s}\chi(\varpi)\,\mathfrak{st}(t_\pi)_{ii}} \\ &= \prod_{i=1}^{7}\left(\sum_{k=0}^{\infty}\left(q^{-s}\chi(\varpi)\,\mathfrak{st}(t_\pi)_{ii}\right)^k\right) = \sum_{k=0}^{\infty} \operatorname{Tr}\left(Sym^k(t_\pi)\right)\chi^k(\varpi)\,q^{-ks}.\end{aligned}$$

Here $\mathfrak{st}(t_\pi)_{ii}$ refers to the (i,i)-th entry of the 7×7 matrix $\mathfrak{st}(t_\pi)$, which is assumed to be diagonal. Plugging Equation (4.2) into the above equality yields

$$\mathcal{L}(s,\pi,\chi,\mathfrak{st})\,v_0 = \sum_{k=0}^{\infty}\left(\int_{G(F)} A_k(g)\,\pi(g)\,v_0 dg\right)\chi^k(\varpi)\,q^{-ks}.$$

For what follows, let us introduce several standard notations. Let $X_*(T) = Hom(T,\mathbb{G}_m) \cong \mathbb{Z}^2$ denote the cocharacter lattice of G and for any $\gamma \in X_*(T)$ denote by $t_\gamma = \gamma(\varpi)$ its representative in the maximal torus T. Let $X_*(T)^+$ denote the set of dominant coweights. There is a partial order on $X_*(T)$: $\gamma \le \lambda$ if and only if $\lambda - \gamma$ can be written as a non-negative combination of the positive coroots. Let ρ be half of the sum of all the positive roots.

Making use of the Cartan decomposition $G = KX_*(T)^+K$ we obtain

$$\sum_{k=0}^{\infty}\left(\int_G A_k(g)\cdot\pi(g)\,v_0 dg\right)\chi^k(\varpi)\,q^{-ks} = \sum_{k=0}^{\infty}\sum_{\gamma\in X_*(T)^+} A_k(t_\gamma)\cdot\omega_\pi(t_\gamma)\lambda(Kt_\gamma K)\,\chi^k(\varpi)\,q^{-ks}v_0,$$

where $\omega_\pi(G) = \langle v_0^\vee, \pi(g)v_0\rangle$ is the normalized spherical function. Here $v_0^\vee$ is the K-fixed vector in $\pi^\vee$ such that $\langle v_0^\vee, v_0\rangle = 1$.

Assuming the double infinite series on the right-hand side is absolutely convergent for $\mathfrak{Re}(s) \gg 0$ it follows that one can change the order of summation. We obtain, by another application of the Cartan decomposition,

$$\mathcal{L}(s,\pi,\chi,\mathfrak{st})\,v_0 = \sum_{k=0}^{\infty}\sum_{\gamma\in X_*(T)^+} A_k(t_\gamma)\cdot\omega_\pi(t_\gamma)\lambda(Kt_\gamma K)\,\chi^k(\varpi)\,q^{-ks}v_0$$

$$= \sum_{\gamma \in X_*(T)^+} \sum_{k=0}^{\infty} A_k(t_\gamma) \cdot \omega_\pi(t_\gamma) \lambda(K t_\gamma K) \chi^k(\varpi) q^{-ks} v_0$$

$$= \int_{G(F)} \left(\sum_{k=0}^{\infty} A_k(g) \chi^k(\varpi) q^{-ks} \right) \pi(g) v_0 dg$$

for $\mathfrak{Re}(s) \gg 0$. The statement then holds for

$$\Delta_{\chi,s} = \sum_{k=0}^{\infty} A_k \chi^k(\varpi) q^{-ks},$$

for any unramified representation π. The uniqueness of the generating function follows from the fact that the action of the spherical functions of unramified representations gives rise to a spectral decomposition of $\mathcal{H}$. Namely, it follows from the Satake isomorphism theorem [40] that $f \in \mathcal{H}$, and hence also for elements in $\mathcal{H}[[q^{-s}]]$, is determined by the set

$$\left\{ \left(\pi, \widehat{f}(\pi) \right) : \pi \text{ unramified representation of } G(F) \right\}.$$

Let us show that this double series converges absolutely for any π and hence it is possible to interchange the order of the summation. Since χ is unitary we may assume that $\chi = \mathbb{1}$. For this purpose we shall provide a bound for each term.

Lemma 4.1.2. *1. For any $j \geq 0$*

$$\left| \left\{ \gamma \in X_*(T)^+ : \gamma \leq [j, 0] \right\} \right| \leq (j+1)(2j+1).$$

2. $A_j(t_\gamma) = 0$ unless $\gamma \leq [j, 0]$.

3. Assume $\gamma \leq [j, 0]$. Then there exist constants $C_1, C_2, C_3, z > 0$ such that

$$\begin{cases} |A_j(t_\gamma)| \leq C_1 j^7 \\ |\omega_\pi(t_\gamma)| \leq C_2 q^{jz} \\ \lambda(K t_\gamma K) \leq C_3 q^{6j} \end{cases}.$$

Proof. For any dominant coweight λ, which can be simultaneously regarded as a dominant weight of the dual group, denote by $A_\lambda \in \mathcal{H}$ the function corresponding to the highest weight irreducible representation V_λ of ${}^L G$ via the Satake isomorphism.

Let $\gamma = n\alpha^\vee + m\beta^\vee$. Then $\gamma \leq [j,0] \Rightarrow n \leq j, m \leq 2j$. Obviously, the number of such γ is bounded by $(j+1)(2j+1)$, this proves *1.*

Recall from [39, p. 836] that the j-symmetric algebra of $\mathfrak{st}$ decomposes as follows

$$Sym^j(\mathfrak{st}) = \bigoplus_{\substack{k=0 \\ k\equiv j (\mathrm{mod}\ 2)}}^{j} V_{[k,0]}.$$

Hence $A_j(g) = \sum_{\substack{k=0 \\ k\equiv j (\mathrm{mod}\ 2)}}^{j} A_{[k,0]}(g)$. According to [23, Section 4]

$$A_\lambda(t_\gamma) = \begin{cases} q^{-(\lambda,\rho)} P_{\lambda,\gamma}(q) & \gamma \leq \lambda \\ 0 & \text{otherwise} \end{cases},$$

where $P_{\lambda,\gamma}$ is an affine Kazhdan-Lusztig polynomial of degree at most $\langle \lambda - \gamma, \rho \rangle$ with non-negative integral coefficients. *2* follows immediately.

We now prove *3.* In particular for $\gamma \leq \lambda$ it holds that

$$0 \leq A_\lambda(t_\gamma) = q^{-(\lambda,\rho)} P_{\lambda,\gamma}(q) \leq P_{\lambda,\gamma}(1) = \dim V_\lambda(\gamma) \leq \dim V_\lambda.$$

Here $V_\lambda(\gamma)$ denotes the γ-eigenspace in V_λ.

By the Weyl character formula [12, Cor. 24.6]

$$\dim V_\lambda = \prod_{\alpha>0} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}.$$

There exists $C_1 > 0$ such that $\dim V_{[k,0]} \leq C_1 k^6$. In particular for $\gamma \leq [j,0]$ it holds that

$$A_j(t_\gamma) = \sum_{\substack{k=0 \\ k=j (\mathrm{mod}\ 2)}}^{j} A_k(t_\gamma) \leq C_1 j^7.$$

This proves the first bound in *3.*

Let us prove the second bound in *3.* Assume that the unramified representation π is a constituent of $\mathrm{Ind}_B^G \chi$ where χ is an unramified character. There exists a constant $z > 0$ such that for any $w \in W$ it holds that

$$|w\chi(h_\alpha(\varpi))|, |w\chi(h_\beta(\varpi))| \leq q^z.$$

For $\gamma = n\alpha^\vee + m\beta^\vee \leq [j,0]$ it follows that $|w\chi(t_\gamma)| \leq q^{jz} \cdot q^{2jz} = q^{3jz}$.

Hence by Macdonald's formula [8, Theorem 4.2] there exists $C_2 > 0, z > 0$ such that

$$|\omega_\pi(t_\gamma)| < C_2 q^{3jz}.$$

Finally, it is known [33, Sec. 3.2] that for $\gamma = n\alpha^\vee + m\beta^\vee$ it holds that $\lambda(Kt_\gamma K) = C_3 q^{2n+2m}$ for some constant $C_3 > 0$. For $\gamma \leq [j,0]$ it is bounded by $C_3 q^{6j}$ and hence *3* is proven. □

Taking all the bounds into account we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty} \sum_{t_\gamma \in X_*(T)^+} |A_j(t_\gamma)| \cdot |\omega_\pi(t_\gamma)| \lambda(Kt_\gamma K) \|v_0\| \left|q^{-js}\right| \\
&\leq \sum_{j=0}^{\infty} \sum_{\gamma \leq [j,0]} |A_j(t_\gamma)| \cdot |\omega_\pi(t_\gamma)| \lambda(Kt_\gamma K) \|v_0\| q^{-j\,\mathfrak{Re}(s)} \\
&\leq \sum_{j=0}^{\infty} \sum_{\gamma \leq [j,0]} C_1 j^7 \cdot C_2 q^{3jz} \cdot C_3 q^{6j} \cdot q^{-j\,\mathfrak{Re}(s)} \|v_0\| \\
&\leq C \cdot \sum_{j=0}^{\infty} j^9 \cdot q^{(6+3z-\mathfrak{Re}(s))j} \|v_0\|
\end{aligned}$$

which converges absolutely for $\mathfrak{Re}(s) \gg 0$.

The bounds in the lemma ensure that for $\mathfrak{Re}(s) \gg 0$ the series $\sum_{j=0}^{\infty} A_j \chi^j(\varpi) q^{-js}$ converges absolutely and uniformly (on G) and hence the function $\Delta_{\chi,s}(g)$ is defined for any s in some right half-plane. □

Furthermore, for any complex character Ψ of $U(F)$ and any $\Lambda \in \mathrm{Hom}_{U(F)}(\pi, \mathbb{C}_\Psi)$ it holds that

$$\begin{aligned}
\mathcal{L}(s,\pi,\chi,\mathfrak{st}) \Lambda(v_0) &= \int_{G(F)} \Delta_{\chi,s}(g) \Lambda(\pi(g) v_0)\, dg \\
&= \int_{U(F)\backslash G(F)} \left(\int_{U(F)} \Delta_{\chi,s}(ug) \Psi(u)\ du \right) \Lambda(\pi(g) v_0)\, dg \qquad (4.4) \\
&= \int_{U(F)\backslash G(F)} \Delta^{\Psi}_{\chi,s}(g) \Lambda(\pi(g) v_0)\, dg,
\end{aligned}$$

where the superscript Ψ notation is defined as in Equation (1.6).

Thus, in order to prove Theorem 2.0.6 it is sufficient to prove

$$\boxed{\Delta^{\Psi_E}_{\chi,s+\frac{1}{2}}(g) = F^*(\Psi_E,\chi,g,s) \quad \forall g \in G(F).} \tag{4.5}$$

We now reduce the proof of Equation (4.5) to the case of $\chi = \mathbb{1}$. For a given unitary character χ of $F^\times$ and $s \in \mathbb{C}$ we denote

$$\chi_s(t) = \chi(t)\,|t|^{s+\frac{5}{2}}. \tag{4.6}$$

We note that in this chapter we use the notation χ_s for a different character than that in Chapter 5 or Chapter 6.

In Section 4.3 we prove the following result.

Proposition 4.1.3. *For $g = h_\alpha(t_1)\,h_\beta(t_2)\,x_\alpha(d) \in M(F)$, let*

$$a_1 = \begin{cases} 1, & \left|\frac{t_2}{t_1^2}a+d\right|_{F_{\alpha_1}} \leq 1 \\ \frac{t_2}{t_1^2}a+d, & \left|\frac{t_2}{t_1^2}a+d\right|_{F_{\alpha_1}} > 1 \end{cases},\ a_2 = \begin{cases} 1, & \left|\frac{t_2}{t_1^2}b+d\right|_{F_{\alpha_3}} \leq 1 \\ \frac{t_2}{t_1^2}b+d, & \left|\frac{t_2}{t_1^2}b+d\right|_{F_{\alpha_3}} > 1 \end{cases},\ a_3 = \begin{cases} 1, & \left|\frac{t_2}{t_1^2}c+d\right|_{F_{\alpha_4}} \leq 1 \\ \frac{t_2}{t_1^2}c+d, & \left|\frac{t_2}{t_1^2}c+d\right|_{F_{\alpha_4}} > 1 \end{cases}.$$

Also denote $\vartheta = \frac{t_1^3}{t_2}a_1a_2a_3 \in F$. It holds that

$$F(\Psi_E,\chi,g,s) = \begin{cases} 0, & |\vartheta|_F > 1 \\ \chi_s\left(\frac{t_2}{\vartheta}\right)\frac{\mathcal{L}\left(s+\frac{3}{2},\chi\right)}{\mathcal{L}\left(s+\frac{5}{2},\chi\right)}\left(|\vartheta| - \chi_s(\varpi\,\vartheta)\,q\right), & |\vartheta|_F \leq 1 \end{cases}.$$

Remark 4.1.4. For g as in Proposition 4.1.3, let $u_1 = \frac{t_2}{t_1^2}a+d$, $u_2 = \frac{t_2}{t_1^2}b+d$ and $u_3 = \frac{t_2}{t_1^2}c+d$ and let $R_g(x,y)$ denote the cubic form corresponding to the character Ψ_E^g. It holds that

$$R_g(x,y) = \frac{t_1^3}{t_2}(x-u_1y)(x-u_2y)(x-u_3y)$$

and $|\vartheta|$ is the supremum of the absolute values of the coefficients of $R_g(x,y)$. In particular, $|\vartheta| \leq 1$ if and only if $R_g(x,y)$ has coefficients in $\mathcal{O}$.

Expanding $F^*(\Psi_E, \chi, g, s)$ as a power series in q^{-s} we get $\sum_{k=0}^{\infty} B_k \chi^k(\varpi) q^{-ks}$, where the coefficients $B_k \in \mathcal{M}_\Psi$ are independent of χ. Hence, in order to prove Equation (4.5), it is enough to prove that

$$\boxed{\Delta^{\Psi_E}_{\mathbb{1},s+\frac{1}{2}}(g) = F^*(\Psi_E, \mathbb{1}, g, s) \quad \forall g \in G(F).} \tag{4.7}$$

We therefore devote the rest of this chapter as well as Appendix A to prove this equality, thus allowing us to drop χ from all notations.

Remark 4.1.5. Since $\Delta^{\Psi_E}_s(g)$ and $F^*(\Psi_E, g, s) \in \mathcal{M}_{\Psi_E}$, it suffices to prove the equality for $g \in M(F) \cap B(F)$. $M_{\Psi_E} l$ was defined in Section 1.6.

While the right-hand side of Equation (4.7) is given explicitly, we do not have an explicit formula for the generating function $\Delta_s(g)$. In [24] we introduced a way to overcome this difficulty.

Recall the Cartan decomposition $G = KT^+K$, where

$$T^+ = \left\{ t \in T : |\gamma(t)| \leq 1 \ \forall \gamma \in \Phi^+ \right\}.$$

Let $D_s \in \mathcal{H}[[q^{-s}]]$ be the bi-K-invariant function defined on T^+ by

$$D_s(t) = |\omega_1(t)|^{s+\frac{7}{2}} \qquad \forall t \in T^+.$$

The function D_s is an *approximation of the generating function* Δ_s in a sense which is apparent from the following proposition [24, Propositions 7.1 and 7.2].

Proposition 4.1.6. *There exist $P_s \in \mathcal{H}[q^{-s}]$ and $s_0 \in \mathbb{R}$ such that for $\mathfrak{Re}\, s > s_0$ it holds that*

$$D_s = \Delta_{s+\frac{1}{2}} * P_s .$$

More precisely

$$P_s = \frac{R_0\left(q^{-s-\frac{1}{2}}\right) A_0 - R_1\left(q^{-s-\frac{1}{2}}\right) A_1}{\xi_F\left(s+\frac{3}{2}\right) \xi_F\left(s+\frac{7}{2}\right) \xi_F\left(s+\frac{1}{2}\right)}, \tag{4.8}$$

where

$$R_0(z) = \frac{z^4}{q^2} + \left(\frac{1}{q^2} + \frac{1}{q}\right) z^3 + \frac{z^2}{q} + \left(\frac{1}{q} + 1\right) z + 1, \quad R_1(z) = \frac{z^2}{q} .$$

Furthermore, there exists s_0 such that for any $\mathfrak{Re}(s) > s_0$ and $f \in \mathcal{M}_{\Psi_E}$

$$f * P_s \equiv 0 \Longrightarrow f \equiv 0.$$

Proof. We first prove the first assertion. Let ${v_0}^\vee \in \pi^\vee$ be the spherical vector such that $\langle {v_0}^\vee, v_0 \rangle = 1$ and let ω_π be the normalized spherical function associated with π given by

$$\omega_\pi(g) = \langle {v_0}^\vee, \pi(g) v_0 \rangle .$$

For any functional l of π it holds that

$$\int_G D_s(g) \, l(\pi(g)v_0) \; dg = l(v_0) \int_G D_s(g) \, \omega_\pi(g) \; dg .$$

Using Macdonald's formula [8, Theorem 4.2] for ω_π this integral turns into a sum of geometric progressions that converges for $\mathfrak{Re}\, s \gg 0$. A straightforward computation yields

$$\widehat{D}_s(\pi) = \int_G D_s(g) \, \omega_\pi(g) \; dg = \mathcal{L}\left(s + \frac{1}{2}, \pi, \mathfrak{st}\right) \cdot Q(\pi, s) \quad . \tag{4.9}$$

Here

$$Q(\pi, s) = \frac{R_0\left(q^{-s-\frac{1}{2}}\right) - R_1\left(q^{-s-\frac{1}{2}}\right) tr(\mathfrak{st})(t_\pi)}{\xi_F\left(s + \frac{3}{2}\right) \xi_F\left(s + \frac{7}{2}\right) \xi_F\left(s + \frac{1}{2}\right)} \quad . \tag{4.10}$$

On the other hand $\mathcal{L}\left(s + \frac{1}{2}, \pi, \mathfrak{st}\right) = \widehat{\Delta}\left(\pi, s + \frac{1}{2}\right)$ and obviously $Q(\pi, s) = \widehat{P}(\pi, s)$.

We now prove the second assertion. Note that $\mathcal{H}$ can be completed into a C*-algebra $\hat{\mathcal{H}}$ as a closed subspace of the reduced group C*-algebra of G. One way to do this is to use the action of $\mathcal{H}$ on $L^2(K \setminus G/K)$ by convolution. This is a separable Hilbert space and $\mathcal{H}$ admits an embedding into the bounded linear operators $\mathcal{B}(L^2(K \setminus G/K))$ on $L^2(K \setminus G/K)$. We consider the completion of $\mathcal{H}$ in $\mathcal{B}(L^2(K \setminus G/K))$ with respect to the operator norm. In fact, for our needs it is enough to know that a C*-norm and such a completion exist.

We need to prove that the operator $*P_s$ is injective for $\mathfrak{Re}(s) \gg 0$. We prove a slightly stronger claim, namely that there exists s_0 such that for any $\mathfrak{Re}(s) > s_0$ the element $P(\cdot, s)$ is invertible in $\hat{\mathcal{H}}$. For $\mathfrak{Re}(s) \gg 0$ this is equivalent to showing that

$$A_0 - \frac{R_1\left(q^{s+\frac{1}{2}}\right)}{R_0\left(q^{s+\frac{1}{2}}\right)} A_1$$

is invertible. Since $\hat{\mathcal{H}}$ is a C*-algebra it will suffice to show that $\left\| \frac{R_1\left(q^{s+\frac{1}{2}}\right)}{R_0\left(q^{s+\frac{1}{2}}\right)} A_1 \right\| < 1$. We have

$$\left\| \frac{R_1\left(q^{s+\frac{1}{2}}\right)}{R_0\left(q^{s+\frac{1}{2}}\right)} A_1 \right\| = \left| \frac{R_1\left(q^{s+\frac{1}{2}}\right)}{R_0\left(q^{s+\frac{1}{2}}\right)} \right| \|A_1\|$$

and since

$$\lim_{\mathfrak{Re}(s)\to\infty} R_0\left(q^{s+\frac{1}{2}}\right) = 1 \quad \text{and} \quad \lim_{\mathfrak{Re}(s)\to\infty} R_1\left(q^{s+\frac{1}{2}}\right) = 0 \ ,$$

there exists s_0 such that for $\mathfrak{Re}(s) > s_0$ we have

$$\left| \frac{R_1\left(q^{s+\frac{1}{2}}\right)}{R_0\left(q^{s+\frac{1}{2}}\right)} \right| < \frac{1}{\|A_1\|} \ .$$

□

Remark 4.1.7. Recall from [23] that $A_0 = \mathbb{1}_K$ and $A_1 = q^{-3}\left(\mathbb{1}_K + \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)$.

Remark 4.1.8. Since the Fourier transform is a map of $\mathcal{H}$-modules, it holds that

$$D_s^{\Psi_E} = \Delta_{s+\frac{1}{2}}^{\Psi_E} * P_s$$

Corollary 4.1.9. *Theorem 2.0.6 follows from*

$$D_s^{\Psi_E} = F^*\left(\Psi_E, \cdot, s\right) * P_s. \tag{4.11}$$

We prove Equation (4.11) by explicitly computing both sides of the equality. Making Equation (4.11) more explicit we prove:

Theorem 4.1.10. *$D_s^{\Psi_E}(g) = 0$ unless $g \in UTK$. Let $g = h_\alpha(t_1)\, h_\beta(t_2)$, where $|t_1| = q^{-n}$ and $|t_2| = q^{-m}$. It holds that*

- *If $N_{a,b,c} = 0$ then*

$$D_s^{\Psi_E}(g) = F^*\left(\Psi_E, \cdot, s\right) * P_s(g) = \begin{cases} \frac{q^{-\left(s+\frac{7}{2}\right)n}}{\xi_F\left(s+\frac{7}{2}\right)}, & \left|\frac{t_1^2}{t_2}\right| = 1 \\ 0, & \left|\frac{t_1^2}{t_2}\right| > 1 \\ \frac{q^{2n-m-\left(s+\frac{7}{2}\right)n}}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)}, & \left|\frac{t_1^2}{t_2}\right| < 1 \end{cases} .$$

- *If* $N_{a,b,c} \in \mathcal{O}^\times$ *then*

$$D_s^{\Psi_E}(g) = F^*(\Psi_E, \cdot, s) * P_s(g) = \begin{cases} \frac{q^{-\left(s+\frac{7}{2}\right)n}}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)}, & \left|\frac{t_1^2}{t_2}\right| = 1 \\ 0, & \left|\frac{t_1^2}{t_2}\right| > 1 \\ 0, & \left|\frac{t_1^2}{t_2}\right| < 1 \end{cases}.$$

This theorem is proved in Sections 4.2 through 4.4 by an explicit calculation of $D_s^{\Psi_E}$ and $F^*(\Psi_E, \cdot, s) * P_s$.

Remark 4.1.11. Equation (4.11) is proved in the split case in [24] for Ψ_s as described there. It can then be deduced for $\Psi_{F\times F\times F}$ by conjugation with $m_{1,-1,0}$ given in Section 3.1. Hence, we devote Sections 4.2 through 4.4 to proving Equation (4.11) assuming that E is non-split. The method is similar to that of the split case, however, some delicate issues arise in the computation.

4.2 Computation of $D_s^{\Psi_E}$

In this section we compute the Ψ_E-Fourier coefficient of D_s. Since Theorem 2.0.6 was already proved for the case $E = F \times F \times F$, we restrict ourselves to the assumption that E is a non-split Galois étale cubic algebra over F.

Theorem 4.2.1. $D_s^{\Psi_E}(g) = 0$ *unless* $g \in UTG(\mathcal{O})$. *If* $g = h_\alpha(t_1) h_\beta(t_2)$, *it holds that*

- *If* $E = F \times K$, *namely* $N_{a,b,c} = 0$, *then*

$$D_s^{\Psi_E}(g) = \begin{cases} \frac{q^{-\left(s+\frac{7}{2}\right)n}}{\xi_F\left(s+\frac{7}{2}\right)}, & \left|\frac{t_1^2}{t_2}\right| = 1 \\ 0, & \left|\frac{t_1^2}{t_2}\right| > 1 \\ \frac{q^{2n-m-\left(s+\frac{7}{2}\right)n}}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)}, & \left|\frac{t_1^2}{t_2}\right| < 1 \end{cases}.$$

- *If E is a field, namely $N_{a,b,c} \in \mathcal{O}^{\times}$, then*

$$D_s^{\Psi_E}(g) = \begin{cases} \frac{q^{-\left(s+\frac{7}{2}\right)n}}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)}, & \left|\frac{t_1^2}{t_2}\right| = 1 \\ 0, & \left|\frac{t_1^2}{t_2}\right| > 1 \\ 0, & \left|\frac{t_1^2}{t_2}\right| < 1 \end{cases} \cdot$$

Before performing a direct computation of $D_s^{\Psi_E}(g)$ we make some preparations. We let SO_7 be the special orthogonal group that preserves the split symmetric form $(\delta_{i,7-i})$ viewed as a subgroup of GL_7. We fix an embedding $\iota : G(F) \hookrightarrow SO_7(F)$ as in [27]. For $h = u(r_1, r_2, r_3, r_4, r_5)\, h_\alpha(t_1) h_\beta(t_2) x_\alpha(d)$ it holds that

$$\iota(h) = \begin{pmatrix} 1 & 0 & r_2 & r_3 & \frac{-r_4}{2} & \frac{r_2r_3+r_5}{2} & \frac{r_2r_4-r_3^2}{2} \\ 0 & 1 & r_1 & r_2 & \frac{-r_3}{2} & \frac{r_1r_3-r_2^2}{2} & \frac{r_1r_4-2r_2r_3-r_5}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{r_3}{2} & \frac{r_4}{2} \\ 0 & 0 & 0 & 1 & 0 & -r_2 & -r_3 \\ 0 & 0 & 0 & 0 & 1 & -r_1 & -r_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t_1 & & & & & & \\ & \frac{t_2}{t_1} & & & & & \\ & & \frac{t_1^2}{t_2} & & & & \\ & & & 1 & & & \\ & & & & \frac{t_2}{t_1^2} & & \\ & & & & & \frac{t_1}{t_2} & \\ & & & & & & \frac{1}{t_1} \end{pmatrix} \cdot \begin{pmatrix} 1 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d & -\frac{d^2}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.12}$$

The following Lemmas 4.2.2 through 4.2.5 are simple to check and will be useful in what follows.

Lemma 4.2.2. *The function $\Gamma : G(F) \to \mathbb{R}$ given by*

$$\Gamma(g) = \max_{1 \le i,j \le 7} |\iota(g)_{i,j}|$$

is a bi-K-invariant function and for $t \in T^+$ it satisfies

$$\Gamma(t) = |\omega_1(t)|^{-1}.$$

Thus, we may write $D_s(g) = \sum_{k=0}^{\infty} \mathcal{D}_k(g) q^{-\left(s+\frac{7}{2}\right)k}$, where

$$\mathcal{D}_k(g) = \begin{cases} 1, & \Gamma(g) = q^k \\ 0, & \text{otherwise} \end{cases}.$$

For any $g \in G$ define $U_k(g) = \left\{u \in U : \Gamma(ug) \leq q^k\right\}$ and let

$$E_k(g) = \begin{cases} 1, & \Gamma(g) \leq q^k \\ 0, & \text{otherwise} \end{cases}.$$

Obviously

$$\mathcal{D}_k(g) = E_k(g) - E_{k-1}(g)$$

and in particular

$$D_s^{\Psi_E}(g) = \sum_{k=0}^{\infty} \left(E_k^{\Psi_E}(g) - E_{k-1}^{\Psi_E}(g)\right) q^{-\left(s+\frac{7}{2}\right)k}.$$

Hence, in order to compute $D_s^{\Psi_E}(g)$ we compute

$$E_k^{\Psi_E}(g) = \int_{U_k(g)} \Psi_E(u)\, du.$$

Lemma 4.2.3. *For $g \in G(F)$ assume that $E_k^{\Psi_E}(g) = 0$ for any $k \neq n, n+1$. Then*

$$D_s^{\Psi_E}(g) = \frac{q^{-\left(s+\frac{7}{2}\right)n}}{\xi_F\left(s+\frac{7}{2}\right)} \left(E_n(g) + E_{n+1}(g)\, q^{-s-\frac{7}{2}}\right).$$

Lemma 4.2.4. *For $n_1, n_2, n_3 \in \mathbb{N}$ with $n_1 + n_2 \geq n_3$ let*

$$\upsilon(n_1, n_2, n_3) = \lambda\left\{(x,y) \in F^2 : |x| \leq q^{n_1}, \quad |y| \leq q^{n_2}, \quad |xy| \leq q^{n_3}\right\}.$$

It holds that

$$\upsilon(n_1, n_2, n_3) = q^{n_3}\left(1 + (n_1 + n_2 - n_3)\left(1 - q^{-1}\right)\right) ,$$

where λ is the Haar measure on F such that $\lambda(\mathcal{O}) = 1$.

Lemma 4.2.5. • $\displaystyle\int_{|x|,|y|,|xy| \leq q} \psi(x+y)\, dx\, dy = -1.$

- *If $E = F \times K$ then*

$$\int\limits_{(\varpi^{-1}\mathcal{O}^{\times})^2} \psi\left(D_{(a,b,c)} r_2 + \frac{r_3^2}{r_2}\right) dr_2\ dr_3 = 1 - q.$$

- *If E is a field then*

$$\int\limits_{(\varpi^{-1}\mathcal{O}^{\times})^2} \psi\left(\frac{r_3^2}{r_2} + D_{(a,b,c)} r_2 - N_{(a,b,c)} \frac{r_2^2}{r_3}\right) dr_2\ dr_3 = 1 - q.$$

The following notation will also be useful in the proof of Theorem 4.2.1, for $n_1, n_2 \in \mathbb{N}$ let

$$\upsilon_{(a,b,c)}^{(n_1)}\left(q^{n_2}\right) = \lambda\left\{(x,y) \in F^2 : |x| = |y| = q^{n_2}, \left|P_{(a,b,c)}\left(\frac{x}{y}\right)\right| \leq q^{n_1 - n_2 \deg\left(P_{(a,b,c)}\right)}\right\},$$

where

$$P_{(a,b,c)}(z) = \begin{cases} z^2 + D_{(a,b,c)}, & N_{(a,b,c)=0} \\ z^3 + D_{(a,b,c)} z - N_{(a,b,c)}, & N_{(a,b,c)\in\mathcal{O}^{\times}} \end{cases}.$$

Let $g = h_\alpha(t_1) h_\beta(t_2) x_\alpha(d) = x_\alpha(p) h_\alpha(t_1) h_\beta(t_2)$, where $p = \frac{dt_1^2}{t_2}$. Denote $|t_1| = q^{-n}$, $|t_2| = q^{-m}$ and $|p| = q^l$. We have $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$\begin{aligned}
&1 \leq q^{k+n} \\
&1, |p| \leq q^{k+m-n} \\
&1, |r_1|, |r_2| \leq q^{k+2n-m} \\
&1, |p|, |r_3 - pr_2|, |r_2 - pr_1| \leq q^k \\
&1, |p|, \left|p^2\right|, \left|2pr_3 - p^2 r_2 - r_4\right|, \left|2pr_2 - r_3 - p^2 r_1\right| \leq q^{k+m-2n} \\
&1, |r_1|, |r_2|, |r_3|, |r_2 r_3 + r_5|, \left|r_2^2 - r_1 r_3\right| \leq q^{k+n-m} \\
&1, |p|, |pr_1 - r_2|, |pr_2 - r_3|, |pr_3 - r_4|, \left|r_2 r_4 - r_3^2 - pr_2 r_3 - pr_5\right| \leq q^{k-n} \\
&\left|r_1 r_4 - 2r_2 r_3 + pr_2^2 - pr_1 r_3 - r_5\right| \leq q^{k-n}.
\end{aligned}$$

4.2.1 Toral elements

In this subsection we consider the case of $g \in UTK$. Since $E_k^{\Psi_E} \in \mathcal{M}_{\Psi_E}$ it is sufficient to consider the case where $g = h_\alpha(t_1) h_\beta(t_2)$. Let $|t_1| = q^{-n}$ and $|t_2| = q^{-m}$.

Remark 4.2.6. From Lemma 1.6.2, we have $D_s^{\Psi_E}(g) = 0$ unless

$$\begin{cases} t_1, t_2, \frac{t_2}{t_1}, \frac{t_1^3}{t_2} \in \mathcal{O}, \quad N_{(a,b,c)} = 0 \\ t_1, t_2, \frac{t_2}{t_1}, \frac{t_1^3}{t_2}, \frac{t_2^2}{t_1^3} \in \mathcal{O}, \ N_{(a,b,c)} \in \mathcal{O}^\times \end{cases}. \tag{4.13}$$

Thus we may assume that $|t_2| \leq |t_1| \leq 1$.

Furthermore, $U_k(g) = \emptyset$ unless $k \geq n, m-n$ and we have $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_3|, \left|r_1 r_3 - r_2^2\right|, |r_2 r_3 + r_5| \leq q^{k+n-m}$$
$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |r_1 r_4 - 2 r_2 r_3 - r_5| \leq q^{k-n}.$$

Lemma 4.2.7. *As an analogue of [24, Lemma B.2], note that*

$$\int_{U_k(g)} \Psi_E(u)\, du = \int_{\widehat{U_k(g)}} \Psi_E(u)\, du,$$

where

$$\widehat{U_k(g)} = \left\{u(r_1, r_2, r_3, r_4, r_5) \in U_k(g) : \left|r_4 + D_{(a,b,c)} r_2 - N_{(a,b,c)} r_1\right| \leq q\right\}.$$

Proof. This follows immediately from the above inequalities and the fact that ψ is of conductor $\mathcal{O}$. For $x \in F$ we write

$$U_k(g, x) = \left\{u(r_1, r_2, r_3, r_4, r_5) : x = r_4 + D_{(a,b,c)} r_2 - N_{(a,b,c)} r_1\right\}.$$

One checks that for any $|x| > q$ and any $\alpha \in \mathcal{O}^\times$ it holds that $U_k(g, x) = U_k(g, \alpha x)$. Hence, it follows that

$$\int_{U_k(g)} \Psi_E(u)\, du = \int_F \psi(x) \int_{U_k(g,x)} du\, dx$$
$$= \int_{\varpi^{-1}\mathcal{O}} \int_{U_k(g,x)} \psi(x) + \sum_{m=1}^{\infty} \int_{U_k(g,\varpi^{-m})} \left(\int_{\varpi^{-m}\mathcal{O}^\times} \psi(x)\, dx \right) du = \int_{\widehat{U_k(g)}} \Psi_E(u)\, du$$

□

We now consider separately the two cases, $N_{(a,b,c)} = 0$ or $N_{(a,b,c)} \in \mathcal{O}^\times$.

4.2.1.1 $N_{(a,b,c)} = 0$

Case of $m < 2n$: We write

$$z = r_1 r_4 - 2r_2 r_3 - r_5.$$

In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\left|r_4 + D_{(a,b,c)} r_2 - N_{(a,b,c)} r_1\right| \le q \qquad |r_1|, \left|r_1 r_3 - r_2^2\right|, |r_1 r_4 - r_2 r_3| \le q^{k+n-m}$$
$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |z| \le q^{k-n}.$$

- $\underline{k = n}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1| \le q^{2n-m}$$
$$|r_2|, |r_3|, |r_4|, |z| \le 1.$$

  Hence

$$E_n^{\Psi_E}(g) = \int\limits_{\varpi^{m-2n}\mathcal{O}} dr_1 \int\limits_{\mathcal{O}^4} \psi\left(r_4 + D_{(a,b,c)} r_2\right) dr_2\, dr_3\, dr_4\, dz = q^{2n-m}.$$

- $\underline{k = n+1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_1 r_3|, |r_1 r_4| \le q^{2n-m+1}$$
$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |z| \le q.$$

  Note that since $|r_2|, |r_3|, |r_4|, |r_2 r_4 - r_3^2|, |z| \le q$, if $|r_3| = q$ then also $|r_2| = |r_4| = q$.

  - $\underline{|r_3|, |r_4| \le 1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1| \le q^{2n-m+1}$$
$$|r_2|, |z| \le q.$$

  - $\underline{|r_3| \le 1, |r_4| = q}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1| \le q^{2n-m}$$
$$|r_2| \le 1$$
$$|z| \le q.$$

– $\underline{|r_3| = q}$: In this case, $u(r_1,r_2,r_3,r_4,r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1| \le q^{2n-m}$$
$$\left|r_2r_4 - r_3^2\right|, |z| \le q.$$

We make a change of variables

$$x = r_2r_4 - r_3^2$$

and then $u(r_1,r_2,r_3,r_4,r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1| \le q^{2n-m}$$
$$|x|, |z| \le q.$$

In conclusion,

$$\begin{aligned}E_{n+1}^{\Psi_E}(g) =& q^{2n-m+2}\int_{\mathcal{O}} \psi(r_4)\, dr_4 \int_{\varpi^{-1}\mathcal{O}} \psi\left(D_{(a,b,c)}r_2\right) dr_2 \\ &+ q^{2n-m+1}\int_{\mathcal{O}} \psi\left(D_{(a,b,c)}r_2\right) dr_2 \int_{\varpi^{-1}\mathcal{O}^\times} \psi(r_4)\, dr_4 \\ &+ q^{2n-m+1}\int_{(\varpi^{-1}\mathcal{O}^\times)^2} \psi\left(D_{(a,b,c)}r_2 + \frac{r_3^2}{r_2}\right) dr_2\ dr_3 = -q^{2n-m+2}.\end{aligned}$$

- $\underline{k > n+1}$: In this case, $u(r_1,r_2,r_3,r_4,r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, \left|r_1r_3 - r_2^2\right|, |r_1r_4 - r_2r_3| \le q^{k+n-m}$$
$$|r_2|, |r_3|, |r_4|, \left|r_2r_4 - r_3^2\right|, |z| \le q^{k-n}.$$

We make a change of variables

$$x = r_4 + D_{(a,b,c)}r_2$$

so that $u(r_1,r_2,r_3,r_4,r_5) \in \widehat{U_k(g)}$ if and only if

$$|x| \le q$$
$$|r_1|, \left|r_1r_3 - r_2^2\right|, \left|r_1\left(x - D_{(a,b,c)}r_2\right) - r_2r_3\right| \le q^{k+n-m}$$
$$|r_2|, |r_3|, \left|r_2\left(x - D_{(a,b,c)}r_2\right) - r_3^2\right|, |z| \le q^{k-n}.$$

- $\underline{|x|, |r_2| \leq 1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_1 r_3| \leq q^{k+n-m}$$
$$|r_3| \leq q^{\frac{k-n}{2}}$$
$$|z| \leq q^{k-n}.$$

- $\underline{|x| \leq 1, 1 < |r_2| \leq q^{\frac{k-n}{2}}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1 r_2|, |r_1 r_3| \leq q^{k+n-m}$$
$$|r_3| \leq q^{\frac{k-n}{2}}$$
$$|z| \leq q^{k-n}.$$

- $\underline{|x| \leq 1, q^{\frac{k-n}{2}} < |r_2| \leq q^{k-n}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, \left|r_1 r_3 - r_2^2\right|, \left|D_{(a,b,c)} r_1 r_2 + r_2 r_3\right| \leq q^{k+n-m}$$
$$|r_3|, \left|D_{(a,b,c)} r_2^2 + r_3^2\right|, |z| \leq q^{k-n}.$$

We make a change of variables

$$y = D_{(a,b,c)} r_1 r_2 + r_2 r_3$$

and then $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|y| \leq q^{k+n-m}$$
$$|r_3| = |r_2|$$
$$\left|D_{(a,b,c)} r_2^2 + r_3^2\right|, |z| \leq q^{k-n}.$$

- $\underline{|x| = q, \left|x - D_{(a,b,c)} r_2\right| \leq 1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_1 r_3| \leq q^{k+n-m}$$
$$|r_3| \leq q^{\frac{k-n}{2}}$$
$$|z| \leq q^{k-n}.$$

- $\underline{|x| = q, \left|x - D_{(a,b,c)} r_2\right| = q}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} &|r_1| \le q^{k+n-m-1} \\ &|r_1 r_3| \le q^{k+n-m} \\ &|r_3| \le q^{\frac{k-n}{2}} \\ &|z| \le q^{k-n}. \end{aligned}$$

- $\underline{|x| = q, q < |r_2| \le q^{\frac{k-n}{2}}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} &|r_1 r_2|, |r_1 r_3| \le q^{k+n-m} \\ &|r_3| \le q^{\frac{k-n}{2}} \\ &|z| \le q^{k-n}. \end{aligned}$$

- $\underline{|x| = q, q^{\frac{k-n}{2}} < |r_2| \le q^{k-n-1}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} &|r_1|, \left|r_1 r_3 - r_2^2\right|, \left|r_1\left(x - D_{(a,b,c)} r_2\right) - r_2 r_3\right| \le q^{k+n-m} \\ &|r_3| = |r_2| \\ &\left|D_{(a,b,c)} r_2^2 + r_3^2\right|, |z| \le q^{k-n}. \end{aligned}$$

We make a change of variables

$$y = r_1\left(x - D_{(a,b,c)} r_2\right) - r_2 r_3$$

and then $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$\begin{aligned} &|y| \le q^{k+n-m} \\ &|r_3| = |r_2| \\ &\left|D_{(a,b,c)} r_2^2 + r_3^2\right|, |z| \le q^{k-n}. \end{aligned}$$

- $\underline{|x| = q, |r_2| = q^{k-n}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, \left|r_1 r_3 - r_2^2\right|, \left|r_1\left(x - D_{(a,b,c)} r_2\right) - r_2 r_3\right| \le q^{k+n-m}$$

$$|r_2|, |r_3|, \left|r_2 \left(x - D_{(a,b,c)} r_2\right) - r_3^2\right|, |z| \leq q^{k-n}.$$

We make a change of variables

$$y = r_1 \left(x - D_{(a,b,c)} r_2\right) - r_2 r_3$$
$$r_2' = r_2 - \frac{x}{2D_{(a,b,c)}}.$$

Note that $|r_3| = |r_2'|$. It holds that $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|y| \leq q^{k+n-m}$$
$$|r_3| = |r_2'|$$
$$\left|D_{(a,b,c)} (r_2')^2 + r_3^2\right|, |z| \leq q^{k-n}.$$

Denote $l = \lfloor \frac{k-n}{2} \rfloor$, it holds that

$$E_k^{\Psi_E}(g) = q^{k-n} \left(\upsilon(k+n-m, l, k+n-m) + \sum_{j=1}^{l} q^j \left(1 - q^{-1}\right) \upsilon(k+n-m-j, l, k+n-m) \right.$$
$$\left. + q^{k+n-m} \sum_{j=l+1}^{k-n} \upsilon_{(a,b,c)}^{(k-n)} \left(q^j\right) \right) - \left(\upsilon(k+n-m, l, k+n-m) \right.$$
$$\left. + \sum_{j=1}^{l} q^j \left(1 - q^{-1}\right) \upsilon(k+n-m-j, l, k+n-m) + q^{k+n-m} \sum_{j=l+1}^{k-n} \upsilon_{(a,b,c)}^{(k+n-m)} \left(q^j\right) \right) = 0.$$

<u>**Case of $m = 2n$:**</u> We write

$$z = r_2 r_3 + r_5$$

In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_2|, |r_3|, |r_4|, |z|, \left|r_1 r_3 - r_2^2\right|, \left|r_2 r_4 - r_3^2\right|, |r_1 r_4 - r_2 r_3| \leq q^{k-n}.$$

We compute $E_k^{\Psi_E}(g)$ for different values of k.

- <u>$k = n$:</u> In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

  $$|r_1|, |r_2|, |r_4|, |x|, |z| \leq 1.$$

  and hence

  $$E_n^{\Psi_E}(g) = \int_{\mathcal{O}^5} \psi\left(r_4 + D_{(a,b,c) r_2}\right) dr_1\, dr_2\, dr_3\, dr_4\, dz = 1.$$

- $\underline{k=n+1}$: In this case, $u(r_1,r_2,r_3,r_4,r_5)\in\widehat{U_k(g)}$ if and only if

$$|r_1|,|r_2|,|r_3|,|r_4|,|z|,\left|r_1r_3-r_2^2\right|,\left|r_2r_4-r_3^2\right|,|r_1r_4-r_2r_3|\leq q.$$

Note that from $|r_1r_3-r_2^2|,|r_2r_4-r_3^2|\leq q$ it follows that $|r_2|=q$ if and only if $|r_3|=q$, in which case $|r_1|=|r_4|=q$ also.

  - $\underline{|r_2|,|r_3|\leq 1}$: In this case, $u(r_1,r_2,r_3,r_4,r_5)\in\widehat{U_k(g)}$ if and only if

$$|r_1|,|r_4|,|z|,|r_1r_4|\leq q.$$

  $\underline{|r_2|,|r_3|=q}$: In this case, $u(r_1,r_2,r_3,r_4,r_5)\in\widehat{U_k(g)}$ if and only if

$$|r_1|,|r_4|,|z|,\left|r_1r_3-r_2^2\right|,\left|r_2r_4-r_3^2\right|,|r_1r_4-r_2r_3|\leq q.$$

  We make change of variables

$$\begin{aligned}x&=r_1r_3-r_2^2\\ y&=r_4r_2-r_3^2.\end{aligned}$$

  and hence $u(r_1,r_2,r_3,r_4,r_5)\in U_k(g)$ if and only if

$$|x|,|y|,|z|\leq q$$

  The other inequalities are satisfied immediately.

In conclusion, we have

$$\begin{aligned}E_{n+1}^{\Psi_E}(g)=&\int\limits_{\varpi^{-1}\mathcal{O}}dz\int\limits_{\mathcal{O}^2}dr_2\;dr_3\int\limits_{|r_1|,|r_4|,|r_1r_4|\leq q}\psi(r_4)\,dr_1\;dr_4\\ &+\int\limits_{\varpi^{-1}\mathcal{O}}dz\int\limits_{\varpi^{-1}\mathcal{O}^\times}dr_3\int\limits_{\varpi^{-1}\mathcal{O}^\times}\psi\left(D_{(a,b,c)}r_2\right)dr_2\int\limits_{\varpi^{-1}\mathcal{O}}\frac{dx}{|r_3|}\int\limits_{\varpi^{-1}\mathcal{O}}\psi\left(\frac{y+r_3^2}{r_2}\right)\frac{dy}{|r_2|}\\ &=q(q-1)+\int\limits_{(\varpi^{-1}\mathcal{O}^\times)^2}\psi\left(D_{(a,b,c)}r_2+\frac{r_3^2}{r_2}\right)dr_2\;dr_3\int\limits_{\varpi^{-1}\mathcal{O}}\psi\left(\frac{y}{r_2}\right)dy=0.\end{aligned}$$

- $\underline{k > n+1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_2|, |r_3|, |r_4|, |z|, \left|r_1 r_3 - r_2^2\right|, \left|r_2 r_4 - r_3^2\right|, |r_1 r_4 - r_2 r_3| \leq q^{k-n}.$$

We make a change of variables

$$x = r_4 + D_{(a,b,c)} r_2.$$

We then have $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|x| \leq q$$
$$|r_1|, |r_2|, |r_3|, |z|, \left|r_1 r_3 - r_2^2\right|, \left|r_2 \left(x - D_{(a,b,c)} r_2\right) - r_3^2\right|, \left|r_1 \left(x - D_{(a,b,c)} r_2\right) - r_2 r_3\right| \leq q^{k-n}.$$

  - $\underline{|x|, |r_2| \leq 1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_3| \leq q^{\frac{k-n}{2}}$$
$$|r_1|, |r_1 r_3|, |z| \leq q^{k-n}.$$

  - $\underline{|x| \leq 1, 1 < |r_2| \leq q^{\frac{k-n}{2}}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_3| \leq q^{\frac{k-n}{2}}$$
$$|z|, |r_1 r_2|, |r_1 r_3| \leq q^{k-n}.$$

  - $\underline{|x| \leq 1, q^{\frac{k-n}{2}} < |r_2| \leq q^{k-n}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_3|, |z|, \left|r_1 r_3 - r_2^2\right|, \left|D_{(a,b,c)} r_2^2 + r_3^2\right|, \left|\left(D_{(a,b,c)} r_1 + r_3\right) r_2\right| \leq q^{k-n}.$$

    Note that from $\left|D_{(a,b,c)} r_2^2 + r_3^2\right| \leq q^{k-n}$ it follows that $|r_2| = |r_3|$. We make a change of variables

$$y = \left(D_{(a,b,c)} r_1 + r_3\right) r_2$$

    to have $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_2| = |r_3|$$
$$|y|, |z|, \left|D_{(a,b,c)} r_2^2 + r_3^2\right| \leq q^{k-n}.$$

– $\underline{|x| = q, \left|x - D_{(a,b,c)}r_2\right| \leq 1}$: In this case $|r_2| \leq q \leq q^{k-n}$. We have $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} &|r_3| \leq q^{\frac{k-n}{2}} \\ &|r_1|, |z|, |r_1 r_3| \leq q^{k-n}. \end{aligned}$$

– $\underline{|x| = q, \left|x - D_{(a,b,c)}r_2\right| = q}$: In this case $|r_2| \leq q \leq q^{k-n}$. We have $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} &|r_1| \leq q^{k-n-1} \\ &|r_3| \leq q^{\frac{k-n}{2}} \\ &|z|, |r_1 r_3| \leq q^{k-n}. \end{aligned}$$

– $\underline{|x| = q, q < |r_2| \leq q^{\frac{k-n}{2}} \leq q^{k-n-1}}$: Note that in this case (if it happens) $\left|x - D_{(a,b,c)}r_2\right| = |r_2|$ and hence $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} &|r_3| \leq q^{\frac{k-n}{2}} \\ &|z|, |r_1 r_3|, |r_1 r_2| \leq q^{k-n}. \end{aligned}$$

– $\underline{|x| = q, q^{\frac{k-n}{2}} < |r_2| \leq q^{k-n-1}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_3|, |z|, \left|r_1 r_3 - r_2^2\right|, \left|D_{(a,b,c)}r_2^2 + r_3^2\right|, \left|r_1\left(x - D_{(a,b,c)}r_2\right) - r_2 r_3\right| \leq q^{k-n}.$$

Note that here $|r_1| = |r_2| = |r_3|$ since $\left|r_1 r_3 - r_2^2\right|, \left|D_{(a,b,c)}r_2^2 + r_3^2\right| \leq q^{k-n}$ and $q^{\frac{k-n}{2}} < |r_2|$. We make a change of variables

$$y = r_1\left(x - D_{(a,b,c)}r_2\right) - r_2 r_3$$

and then $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$\begin{aligned} &|r_2| = |r_3| \\ &|y|, |z|, \left|D_{(a,b,c)}r_2^2 + r_3^2\right| \leq q^{k-n}. \end{aligned}$$

– <u>$|x| = q, |r_2| = q^{k-n}$</u>: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_3|, |z|, \left|r_1 r_3 - r_2^2\right|, \left|r_2\left(x - D_{(a,b,c)} r_2\right) - r_3^2\right|, \left|r_1\left(x - D_{(a,b,c)} r_2\right) - r_2 r_3\right| \le q^{k-n}.$$

Note that here $|r_1| = |r_2| = |r_3| = q^{k-n}$ since $\left|r_1 r_3 - r_2^2\right|, \left|D_{(a,b,c)} r_2^2 + r_3^2\right| \le q^{k-n}$ and $q^{\frac{k-n}{2}} < |r_2|$. We make a change of variables

$$y = r_1\left(x - D_{(a,b,c)} r_2\right) - r_2 r_3$$
$$r_2' = r_2 - \frac{x}{2D_{(a,b,c)}}.$$

Note that $|r_2| = |r_2'|$. We then have $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|y|, |r_3|, |z|, \left|D_{(a,b,c)}\left(r_2'\right)^2 + r_3^2\right| \le q^{k-n}.$$

Denoting $l = \lfloor \frac{k-n}{2} \rfloor$, it holds that

$$E_k^{\Psi_E}(g) = q^{k-n}\Bigg(\upsilon(k-n, l, k-n) + \sum_{j=1}^{l} q^j\left(1 - q^{-1}\right)\upsilon(k-n-j, l, k-n)$$
$$+ q^{k-n}\sum_{j=l+1}^{k-n} \upsilon_{(a,b,c)}^{(k-n)}\left(q^j\right)\Bigg) - \Big(\upsilon(k-n, l, k-n) + q\left(1 - q^{-1}\right)\upsilon(k-n-1, l, k-n)$$
$$+ \sum_{j=2}^{l} q^j\left(1 - q^{-1}\right)\upsilon(k-n-j, l, k-n) + q^{k-n}\sum_{j=l+1}^{k-n} \upsilon_{(a,b,c)}^{(k-n)}\left(q^j\right)\Big) = 0.$$

<u>Case of $m > 2n$</u>: Since the proof of this case is identical to the case of $N_{(a,b,c)} \in \mathcal{O}^\times$, we write here

$$\Psi_E\left(u(r_1, r_2, r_3, r_4, r_5)\right) = \psi\left(r_4 + D_{(a,b,c)} r_2 - N_{(a,b,c)} r_1\right)$$

so that the proof fits both cases. We write

$$z = r_2 r_3 + r_5$$

In this case, $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$|r_1|, |r_2|, |r_3|, \left|r_1 r_3 - r_2^2\right|, |z| \le q^{k+n-m}$$
$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |r_1 r_4 - r_2 r_3| \le q^{k-n}.$$

- $\underline{k = m - n}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_2|, |r_3|, |z| \leq 1$$
$$|r_4| \leq q^{m-2n}.$$

Since $m - 2n \geq 1$ it holds that

$$E_n^{\Psi_E}(g) = \int\limits_{|r_4| \leq q^{m-2n}} \psi(r_4)\, dr_4 = 0.$$

- $\underline{k = m - n + 1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_2|, |r_3|, \left|r_1 r_3 - r_2^2\right|, |z| \leq q$$
$$|r_4|, |r_2 r_4|, |r_1 r_4| \leq q^{m-2n+1}.$$

  - $\underline{|r_1|, |r_2| \leq 1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_3|, |z| \leq q$$
$$|r_4| \leq q^{m-2n+1}.$$

  - $\underline{|r_2| \leq 1, |r_1| = q}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_3| \leq 1$$
$$|z| \leq q$$
$$|r_4| \leq q^{m-2n}.$$

  - $\underline{|r_2| = q}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1| = |r_3| = q$$
$$|z| \leq q$$
$$|r_4| \leq q^{m-2n}.$$

In conclusion, we have

$$E_{n+1}^{\Psi_E}(g) = q^2 \int\limits_{|r_4| \leq q^{m-2n+1}} \psi(r_4)\, dr_4 + q(q-1) \int\limits_{|r_4| \leq q^{m-2n}} \psi(r_4)\, dr_4$$
$$+ q(q-1) \int\limits_{|r_1| = |r_2| = q} \psi\left(D_{(a,b,c)} r_2 - N_{(a,b,c)} r_1\right) dr_1\, dr_2\, dr_4 \int\limits_{|r_4| \leq q^{m-2n}} \psi(r_4)\, dr_4 = 0.$$

- <u>$k > m - n + 1$:</u> In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_2|, |r_3|, \left|r_1 r_3 - r_2^2\right|, |z| \leq q^{k+n-m}$$
$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |r_1 r_4 - r_2 r_3| \leq q^{k-n}.$$

We make a change of variables

$$x = r_4 + D_{(a,b,c)} r_2 - N_{(a,b,c)} r_1$$

and then $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|x| \leq q$$
$$|r_1|, |r_2|, |r_3|, \left|r_1 r_3 - r_2^2\right|, |z| \leq q^{k+n-m}$$
$$\left|r_2\left(N_{(a,b,c)} r_1 - D_{(a,b,c)} r_2\right) - r_3^2\right|, \left|r_1\left(N_{(a,b,c)} r_1 - D_{(a,b,c)} r_2\right) - r_2 r_3\right| \leq q^{k-n}.$$

Denote by $C \subset F^3$ the set of elements $(r_1, r_2, r_3) \in C$ such that

$$|r_1|, |r_2|, |r_3|, \left|r_1 r_3 - r_2^2\right| \leq q^{k+n-m}$$
$$\left|r_2\left(N_{(a,b,c)} r_1 - D_{(a,b,c)} r_2\right) - r_3^2\right|, \left|r_1\left(N_{(a,b,c)} r_1 - D_{(a,b,c)} r_2\right) - r_2 r_3\right| \leq q^{k-n}.$$

It then holds that

$$E_k^{\Psi_E}(g) = q^{k+n-m} \lambda(C) \int\limits_{|x| \leq q} \psi(x)\, dx = 0.$$

4.2.1.2 $N_{(a,b,c)} \in \mathcal{O}^\times$

<u>Case of $m < 2n$</u>: We write

$$z = r_1 r_4 - 2 r_2 r_3 - r_5.$$

In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, \left|r_1 r_3 - r_2^2\right|, |r_1 r_4 - r_2 r_3| \leq q^{k+n-m}$$
$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q^{k-n}.$$

- $\underline{k = n}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1| \leq q^{2n-m}$$
$$|r_2|, |r_3|, |r_4|, |z| \leq 1$$

and hence

$$E_n^{\Psi_E}(g) = \int\limits_{|r_1| \leq q^{2n-m}} \psi\left(-N_{(a,b,c)} r_1\right) dr_1 = 0,$$

since $2n - m \geq 1$.

- $\underline{k = n+1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_1 r_3|, |r_1 r_4| \leq q^{2n-m+1}$$
$$|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q.$$

  - $\underline{|r_4| \leq 1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1| \leq q^{2n-m+1}$$
$$|r_2|, |z| \leq q$$
$$|r_3| \leq 1.$$

  - $\underline{|r_4| = q}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1| \leq q^{2n-m}$$
$$|r_2|, |r_3|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q.$$

In conclusion,

$$E_{n+1}^{\Psi_E}(g) = q^2 \int\limits_{|r_1| \leq q^{2n-m+1}} \psi\left(-N_{(a,b,c)} r_1\right) dr_1$$
$$+ q \int\limits_{|r_1| \leq q^{2n-m}} \psi\left(-N_{(a,b,c)} r_1\right) dr_1 \int\limits_{\substack{|r_4| = q \\ |r_2|, |r_3|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q}} \psi\left(r_4 + D_{(a,b,c)} r_2\right) dr_2\, dr_3\, dr_4 = 0.$$

- $\underline{k > n+1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} &|r_1|, \left|r_1 r_3 - r_2^2\right|, |r_1 r_4 - r_2 r_3| \leq q^{k+n-m} \\ &|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q^{k-n}. \end{aligned}$$

We make a change of variables

$$x = r_4 + D_{(a,b,c)} r_2 - N_{(a,b,c)} r_1$$

and then $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} &|x| \leq q \\ &\left|\left(r_4 + D_{(a,b,c)} r_2\right) r_3 - N_{(a,b,c)} r_2^2\right|, \left|\left(r_4 + D_{(a,b,c)} r_2\right) r_4 - N_{(a,b,c) r_2 r_3}\right| l \leq q^{k+n-m} \\ &|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |z| \leq q^{k-n}. \end{aligned}$$

We denote by $C \subset F^3$ the set of all elements (r_2, r_3, r_4) such that

$$\begin{aligned} &\left|\left(r_4 + D_{(a,b,c)} r_2\right) r_3 - N_{(a,b,c)} r_2^2\right|, \left|\left(r_4 + D_{(a,b,c)} r_2\right) r_4 - N_{(a,b,c) r_2 r_3}\right| \leq q^{k+n-m} \\ &|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right| \leq q^{k-n}. \end{aligned}$$

It holds that

$$E_k^{\Psi_E}(g) = q^{k-n} \lambda(C) \int_{|x| \leq q} \psi(x)\, dx = 0.$$

Case of $m = 2n$: We write

$$z = r_2 r_3 + r_5.$$

In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_2|, |r_3|, |r_4|, |z|, \left|r_1 r_3 - r_2^2\right|, \left|r_2 r_4 - r_3^2\right|, |r_1 r_4 - r_2 r_3| \leq q^{k-n}.$$

- $\underline{k = n}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_2|, |r_3|, |r_4|, |z| \leq 1$$

and hence

$$E_n^{\Psi_E}(g) = 1.$$

- $\underline{k = n+1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_2|, |r_3|, |r_4|, |z|, \left|r_1 r_3 - r_2^2\right|, \left|r_2 r_4 - r_3^2\right|, |r_1 r_4 - r_2 r_3| \leq q.$$

Note that $|r_2| = q$ if and only if $|r_3| = q$, in which case $|r_1| = |r_4| = q$.

  - $\underline{|r_2|, |r_3| \leq 1}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_4|, |z|, |r_1 r_4| \leq q.$$

  $\underline{|r_2| = |r_3| = q}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1|, |r_4|, |z|, \left|r_1 r_3 - r_2^2\right|, \left|r_2 r_4 - r_3^2\right|, |r_1 r_4 - r_2 r_3| \leq q.$$

  We make a change of variables

$$\begin{aligned} x &= r_1 r_3 - r_2^2 \\ y &= r_2 r_4 - r_3^2. \end{aligned}$$

  It then holds that $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|x|, |y|, |z| \leq q.$$

In conclusion

$$\begin{aligned} E_{n+1}^{\Psi_E}(g) = q & \int\limits_{|r_1|,|r_4|,|r_1 r_4| \leq q} \psi\left(r_4 + D_{(a,b,c)} r_2\right) dr_2\ dr_4 \\ & + q \int\limits_{\substack{|r_2|=|r_3|=q \\ |x|,|y|\leq q}} \psi\left(\frac{y + r_3^2}{r_2} + D_{(a,b,c)} r_2 - N_{(a,b,c)} \frac{x + r_2^2}{r_3}\right) dr_2\ dr_3\ \frac{dx}{q}\ \frac{dy}{q} \\ & = -q + q(1-q) = -q^2. \end{aligned}$$

- $\underline{k > n+1}$: We make a change of variables

$$x = r_4 + D_{(a,b,c)} r_2 - N_{(a,b,c)} r_1.$$

In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} & |x| \leq q \\ & |r_1|, |r_2|, |r_3|, |z|, \left|r_1\left(x - D_{(a,b,c)} r_2 + N_{(a,b,c)} r_1\right) - r_2 r_3\right| \leq q^{k-n} \\ & \left|r_1 r_3 - r_2^2\right|, \left|r_2\left(x - D_{(a,b,c)} r_2 + N_{(a,b,c)} r_1\right) - r_3^2\right| \leq q^{k-n}. \end{aligned}$$

- $\underline{|r_2| \leq q^{\frac{k-n}{2}}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} &|x| \leq q \\ &|r_1|, |r_3| \leq q^{\frac{k-n}{2}} \\ &|z| \leq q^{k-n}. \end{aligned}$$

- $\underline{q^{\frac{k-n}{2}} < |r_2| < q^{k-n}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} &|x| \leq q \\ &|r_1| \leq q^{k-n-1} \\ &|r_3|, |z|, \left|r_1\left(N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right) - r_2 r_3\right| \leq q^{k-n} \\ &\left|r_1 r_3 - r_2^2\right|, \left|r_2\left(N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right) - r_3^2\right| \leq q^{k-n}. \end{aligned}$$

- $\underline{|r_2| = q^{k-n}}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} &|x| \leq q \\ &|r_1|, |r_2|, |r_3|, |z|, \left|r_1\left(x - D_{(a,b,c)}r_2 + N_{(a,b,c)}r_1\right) - r_2 r_3\right| \leq q^{k-n} \\ &\left|r_1 r_3 - r_2^2\right|, \left|r_2\left(x - D_{(a,b,c)}r_2 + N_{(a,b,c)}r_1\right) - r_3^2\right| \leq q^{k-n}. \end{aligned}$$

  First note that $|r_1| = |r_2| = |r_3| = q^{k-n}$, hence we may write $\epsilon = \frac{r_2}{r_1}$ with $|\epsilon| = 1$. Consider $\left|r_1\left(x - D_{(a,b,c)}r_2 + N_{(a,b,c)}r_1\right) - r_2 r_3\right| \leq q^{k-n}$, multiplying by ϵ^3 and dividing by r_2^2 we have

$$\left|\left(\epsilon^3 + D_{(a,b,c)}\epsilon - N_{(a,b,c)}\right) - \frac{\epsilon x}{r_2}\right| \leq \frac{1}{q^{k-n}} < 1.$$

  Since E/F is unramified and $\epsilon \in \mathcal{O}$, it holds that $\left|\epsilon^3 + D_{(a,b,c)}\epsilon - N_{(a,b,c)}\right| = 1$ and hence also $\left|\frac{\epsilon x}{r_2}\right| = 1$, contradicting the fact that $\left|\frac{\epsilon x}{r_2}\right| \leq \frac{q}{q^{k-n}} < 1$.

Let $l = \lfloor \frac{k-n}{2} \rfloor$. Also, let $C \subset F^3$ be the set such that $(r_1, r_2, r_3) \in C$ if and only if

$$\begin{aligned} &q^{\frac{k-n}{2}} < |r_2| \leq q^{k-n-1} \\ &|r_1| \leq q^{k-n-1} \end{aligned}$$

$$\begin{aligned}&\left|r_3\right|, \left|r_1\left(N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right) - r_2r_3\right| \leq q^{k-n}\\&\left|r_1r_3 - r_2^2\right|, \left|r_2\left(N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right) - r_3^2\right| \leq q^{k-n}.\end{aligned}$$

It then holds that

$$E_k^{\Psi_E}(g) = q^{k-n}\left(q^{l+k-n} + \lambda(C)\right)\int\limits_{|x|\leq q}\psi(x)\,dx = 0.$$

<u>**Case of $m > 2n$**</u>**:** This was already dealt with in the discussion of the case $N_{(a,b,c)} = 0$.

4.2.2 Non-Toral Elements - Part I

In the remainder of this section we consider the case of $g \notin UTK$. Since $E_k^{\Psi_E} \in \mathcal{M}_{\Psi_E}$ it is sufficient to consider $g = h_\alpha(t_1)h_\beta(t_2)x_\alpha(d) = x_\alpha(p)\,h_\alpha(t_1)h_\beta(t_2)$, where $p = \frac{t_1^2 d}{t_2}$. Let $|t_1| = q^{-n}$, $|t_2| = q^{-m}$ and $|p| = q^l$. In this subsection we consider the case where $|p| \leq 1$; the case where $|p| > 1$ is discussed in the following subsection. Also note that, under this assumption, it follows that $\left|\frac{t_1^2}{t_2}\right| < 1$.

Remark 4.2.8. From Lemma 1.6.2, and under the assumption that $|p| \leq 1$, we have $D_s^{\Psi_E}(g) = 0$ unless

$$\frac{t_1^3}{t_2}, t_1, \frac{t_2}{t_1^3} \in \mathcal{O}.$$

Furthermore, $U_k(g) = \emptyset$ unless $k \geq n$. We have $u(r_1, r_2, r_3, r_4, r_5) \in U_k(g)$ if and only if

$$\begin{aligned}&k \geq n, m-n\\&\left|r_1\right|, \left|r_2\right|, \left|r_3\right|, \left|r_2r_3 + r_5\right|, \left|r_2^2 - r_1r_3\right| \leq q^{k+n-m}\\&\left|pr_1 - r_2\right|, \left|pr_2 - r_3\right|, \left|pr_3 - r_4\right|, \left|r_2r_4 - r_3^2 - pr_2r_3 - pr_5\right| \leq q^{k-n}\\&\left|r_1r_4 - 2r_2r_3 + pr_2^2 - pr_1r_3 - r_5\right| \leq q^{k-n}.\end{aligned}$$

Remark 4.2.9. As an analogue of [24, Lemma B.2], note that

$$\int\limits_{U_k(g)}\Psi_E(u)\,du = \int\limits_{\widehat{U_k(g)}}\Psi_E(u)\,du,$$

where

$$\widehat{U_k(g)} = \left\{u(r_1, r_2, r_3, r_4, r_5) \in U_k(g) : \left|r_4 + D_{(a,b,c)}r_2 - N_{(a,b,c)}r_1\right| \leq q\right\}.$$

We now split the computation into two cases, $N_{(a,b,c)} = 0$ or $N_{(a,b,c)} \in \mathcal{O}^\times$.

4.2.2.1 $N_{(a,b,c)} = 0$

We write

$$x = r_4 + D_{(a,b,c)} r_2$$
$$z = r_2 r_3 + r_5.$$

In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|x| \le q$$
$$|r_1|, |r_2|, |r_3|, |z|, \left|r_2^2 - r_1 r_3\right| \le q^{k+n-m}$$
$$|pr_1 - r_2|, |pr_2 - r_3|, \left|pr_3 - x + D_{(a,b,c)} r_2\right|, \left|r_2 x - D_{(a,b,c)} r_2^2 - r_3^2 - pz\right| \le q^{k-n}$$
$$\left|r_1 x - D_{(a,b,c)} r_1 r_2 - r_2 r_3 + pr_2^2 - pr_1 r_3 - z\right| \le q^{k-n}.$$

Let $\kappa = p^2 + D_{(a,b,c)}$.

Case of $l < 0$:

- $\underline{k = n}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|x| \le q$$
$$|r_1|, |r_2|, |r_3|, |z|, \left|r_2^2 - r_1 r_3\right| \le q^{2n-m}$$
$$|pr_1 - r_2|, |pr_2 - r_3|, \left|pr_3 - x + D_{(a,b,c)} r_2\right|, \left|r_2 x - D_{(a,b,c)} r_2^2 - r_3^2 - pz\right| \le 1$$
$$\left|r_1 x - D_{(a,b,c)} r_1 r_2 - r_2 r_3 + pr_2^2 - pr_1 r_3 - z\right| \le 1.$$

  - $\underline{|x| \le 1}$: We make a change of variables

$$y = r_1 x - D_{(a,b,c)} r_1 r_2 - z$$

    and then $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|r_1| \le q^{-l}$$
$$|r_2|, |r_3|, |y| \le 1.$$

– $\underline{|x| = q}$: We make a change of variables

$$\begin{aligned} y &= r_1 x - D_{(a,b,c)} r_1 r_2 - r_2 r_3 + p r_2^2 - p r_1 r_3 - z \\ l_1 &= p r_1 - r_2 \\ l_2 &= D_{(a,b,c)} r_2 - x \end{aligned}$$

and then $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|l_1|, |l_2|, |r_3|, |y| \leq 1.$$

In conclusion

$$E_n^{\Psi_E}(g) = q^{-l} \int\limits_{|x| \leq 1} \psi(x)\, dx + q^{-l} \int\limits_{|x| = q} \psi(x)\, dx = 0.$$

- $\underline{k > n}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} & |x| \leq q \\ & |z|, \left| r_2^2 - r_1 r_3 \right| \leq q^{k+n-m} \\ & |r_1|, |r_2|, |r_3|, \left| r_2 x - D_{(a,b,c)} r_2^2 - r_3^2 - p z \right| \leq q^{k-n} \\ & \left| r_1 x - D_{(a,b,c)} r_1 r_2 - r_2 r_3 + p r_2^2 - p r_1 r_3 - z \right| \leq q^{k-n}. \end{aligned}$$

We make a change of variables

$$\begin{aligned} y &= r_1 x - D_{(a,b,c)} r_1 r_2 - r_2 r_3 + p r_2^2 - p r_1 r_3 - z \\ l_1 &= r_2 - p r_1 \\ l_2 &= p r_2 - r_3 \\ l_3 &= D_{(a,b,c)} r_2 + p r_3 - x \end{aligned}$$

and then $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned} & |x| \leq q \\ & |p(p l_2 + l_3) \cdot l_2 + (\kappa l_1 + p l_2 + l_3) l_3| \leq q^{k+n-m+l} \\ & \left| p(p l_2 + l_3)^2 - (\kappa l_1 + p l_2 + l_3)\left(p l_3 - D_{(a,b,c)} l_2\right) \right| \leq q^{k+n-m+l} \end{aligned}$$

$$\left|l_1\right|, \left|pl_2+l_3\right|, \left|pl_3-D_{(a,b,c)}l_2\right|, \left|y\right|, \left|l_1l_3+l_2^2\right| \le q^{k-n}.$$

Let $C \subseteq F^3$ be the subset such that $(l_1,l_2,l_3) \in C$ if and only if

$$\begin{aligned}
&\left|p\left(pl_2+l_3\right)\cdot l_2+\left(\kappa l_1+pl_2+l_3\right)l_3\right| \le q^{k+n-m+l}\\
&\left|p\left(pl_2+l_3\right)^2-\left(\kappa l_1+pl_2+l_3\right)\left(pl_3-D_{(a,b,c)}l_2\right)\right| \le q^{k+n-m+l}\\
&\left|l_1\right|, \left|pl_2+l_3\right|, \left|pl_3-D_{(a,b,c)}l_2\right|, \left|l_1l_3+l_2^2\right| \le q^{k-n}.
\end{aligned}$$

Hence

$$E_k^{\Psi_E}(g) = q^{k-n}\lambda(C)\int\limits_{|x|\le q}\psi(x)\,dx = 0.$$

Case of $l=0$:

- $\underline{k=n}$: We make a change of variables

$$\begin{aligned}
y &= r_1x-D_{(a,b,c)}r_1r_2-r_2r_3+pr_2^2-pr_1r_3-z\\
l_1 &= pr_1-r_2\\
l_2 &= \kappa r_2-x\\
l_3 &= r_3-pr_2.
\end{aligned}$$

In this case, $u(r_1,r_2,r_3,r_4,r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned}
&|x| \le q\\
&|y|, |l_1|, |l_2|, |l_3| \le 1
\end{aligned}$$

and hence

$$E_n^{\Psi_E}(g) = \int\limits_{|x|\le q}\psi(x)\,dx = 0.$$

- $\underline{k>n}$: We make a change of variables

$$\begin{aligned}
y &= r_1x-D_{(a,b,c)}r_1r_2-r_2r_3+pr_2^2-pr_1r_3-z\\
l_1 &= r_2-pr_1\\
l_2 &= pr_2-r_3
\end{aligned}$$

$$l_3 = D_{(a,b,c)} r_2 + p r_3 - x.$$

In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned}
&|x| \le q \\
&\left| \frac{pl_2 + l_3}{\kappa} \cdot l_2 + \left(\frac{pl_2 + l_3}{p\kappa} - \frac{l_1}{p} \right) \cdot l_3 \right| \le q^{k+n-m} \\
&\left| \left(\frac{pl_2 + l_3}{\kappa} \right)^2 - \left(\frac{pl_2 + l_3}{p\kappa} - \frac{l_1}{p} \right) \cdot \frac{pl_3 - D_{(a,b,c)} l_2}{\kappa} \right| \le q^{k+n-m} \\
&\left| \frac{pl_2 + l_3}{p\kappa} - \frac{l_1}{p} \right|, \left| \frac{pl_2 + l_3}{\kappa} \right|, \left| \frac{pl_3 - D_{(a,b,c)} l_2}{\kappa} \right|, \left| l_1 l_3 + l_2^2 \right|, |y| \le q^{k-n}.
\end{aligned}$$

Let $C \subseteq F^3$ be the subset such that $(l_1, l_2, l_3) \in C$ if and only if

$$\begin{aligned}
&\left| \frac{pl_2 + l_3}{\kappa} \cdot l_2 + \left(\frac{pl_2 + l_3}{p\kappa} - \frac{l_1}{p} \right) \cdot l_3 \right| \le q^{k+n-m} \\
&\left| \left(\frac{pl_2 + l_3}{\kappa} \right)^2 - \left(\frac{pl_2 + l_3}{p\kappa} - \frac{l_1}{p} \right) \cdot \frac{pl_3 - D_{(a,b,c)} l_2}{\kappa} \right| \le q^{k+n-m} \\
&\left| \frac{pl_2 + l_3}{p\kappa} - \frac{l_1}{p} \right|, \left| \frac{pl_2 + l_3}{\kappa} \right|, \left| \frac{pl_3 - D_{(a,b,c)} l_2}{\kappa} \right|, \left| l_1 l_3 + l_2^2 \right| \le q^{k-n}.
\end{aligned}$$

Hence

$$E_k^{\Psi_E}(g) = q^{k-n} \lambda(C) \int\limits_{|x| \le q} \psi(x)\, dx = 0.$$

4.2.2.2 $N_{(a,b,c)} \in \mathcal{O}^\times$

We write

$$\begin{aligned}
x &= r_4 + D_{(a,b,c)} r_2 - N_{(a,b,c)} r_1 \\
z &= r_1 r_4 - 2 r_2 r_3 - r_5.
\end{aligned}$$

For any $k \ge n$, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned}
&|x| \le q \\
&|r_1|, |r_2|, |r_3|, |z|, \left| r_2^2 - r_1 r_3 \right| \le q^{k+n-m}
\end{aligned}$$

$$
\begin{aligned}
&|pr_1 - r_2|, |pr_2 - r_3|, \left|pr_3 - \left(x + N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right)\right| \leq q^{k-n} \\
&\left|r_2\left(x + N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right) - r_3^2 - pz\right| \leq q^{k-n} \\
&\left|r_1\left(x + N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right) - r_2r_3 + pr_2^2 - pr_1r_3 - z\right| \leq q^{k-n}.
\end{aligned}
$$

Let $\kappa = p^3 + D_{(a,b,c)}p - N_{(a,b,c)} \in \mathcal{O}^\times$.

- $\underline{k = n}$: We note that

$$
\kappa r_1 - x = p^2\left(pr_1 - r_2\right) + p\left(pr_2 - r_3\right) + \left(pr_3 - \left(x + N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2\right)\right).
$$

Which means that $|\kappa r_1 - x| \leq 1$ and hence $|r_1|, |r_2|, |r_3| \leq q$. We make a change of variables

$$
\begin{aligned}
y &= r_1\left(x + N_{(a,b,c)}r_1 - D_{(a,b,c)}r_2 - pr_3\right) - r_2\left(pr_2 - r_3\right) - z \\
l_1 &= \kappa r_1 - x \\
l_2 &= r_2 - pr_1 \\
l_3 &= r_3 - pr_2
\end{aligned}
$$

and then $u\left(r_1, r_2, r_3, r_4, r_5\right) \in \widehat{U_k\left(g\right)}$ if and only if

$$
\begin{aligned}
&|x| \leq q \\
&\left|\kappa^2 l_2^2 + p\kappa l_1 l_2 - \kappa l_1 l_3\right| \leq q^{2n-m} \\
&|l_1|, |l_2|, |l_3|, |y| \leq 1.
\end{aligned}
$$

Let $C \subseteq F^3$ be the subset such that $(l_1, l_2, l_3) \in C$ if and only if

$$
\begin{aligned}
&\left|\kappa^2 l_2^2 + p\kappa l_1 l_2 - \kappa l_1 l_3\right| \leq q^{2n-m} \\
&|l_1|, |l_2|, |l_3| \leq 1.
\end{aligned}
$$

Hence

$$
E_n^{\Psi_E}\left(g\right) = \lambda\left(C\right) \int\limits_{|x| \leq q} \psi\left(x\right) dx = 0.
$$

- $\underline{k > n}$: We note that

$$\kappa r_1 - x = p^2 (pr_1 - r_2) + p (pr_2 - r_3) + \left(pr_3 - \left(x + N_{(a,b,c)} r_1 - D_{(a,b,c)} r_2\right)\right)$$

Which means that $|\kappa r_1 - x| \leq q^{k-n}$ and hence $|r_1|, |r_2|, |r_3| \leq q^{k-n}$. Moreover $\left|br_3 - \left(x + N_{(a,b,c)} r_1 - D_{(a,b,c)} r_2\right)\right| \leq q^{k-n}$ if and only if $|\kappa r_1 - x| \leq q^{k-n}$.

We make a change of variables

$$\begin{aligned} y &= r_1 \left(x - N_{(a,b,c)} r_1 - D_{(a,b,c)} r_2 - pr_3\right) - r_2 (pr_2 - r_3) - z \\ l_1 &= \kappa r_1 - x \\ l_2 &= r_2 - pr_1 \\ l_3 &= r_3 - pr_2 \end{aligned}$$

and then $u (r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k (g)}$ if and only if

$$\begin{aligned} &|x| \leq q \\ &\left|(\kappa l_2 + pl_1)^2 - l_1 \left(\kappa l_3 + p\kappa l_2 + p^2 l_1\right)\right| \leq q^{k+n-m} \\ &\left|l_1 \left(l_1 + \left(p^2 + D_{(a,b,c)}\right) l_2 + pl_3\right) + (\kappa l_2 + pl_1) l_3\right| \leq q^{k+n-m} \\ &|l_1|, |l_2|, |l_3|, |y|, \left|l_1 \left(l_1 + \left(p^2 + D_{(a,b,c)}\right) l_2 + pl_3\right) - l_3^2\right| \leq q^{k-n}. \end{aligned}$$

Let $C \subseteq F^3$ be the subset such that $(l_1, l_2, l_3) \in C$ if and only if

$$\begin{aligned} &\left|l_1 \left(l_1 + \left(p^2 + D_{(a,b,c)}\right) l_2 + pl_3\right) + (\kappa l_2 + pl_1) l_3\right| \leq q^{k+n-m} \\ &|l_1|, |l_2|, |l_3|, \left|l_1 \left(l_1 + \left(p^2 + D_{(a,b,c)}\right) l_2 + pl_3\right) - l_3^2\right| \leq q^{k-n}. \end{aligned}$$

Hence

$$E_k^{\Psi_E} (g) = q^{k-n} \lambda (C) \int\limits_{|x| \leq q} \psi (x)\, dx = 0.$$

4.2.3 Non-Toral Elements - Part II

We now compute $E_k^{\Psi_E} (g)$ for $g = h_\alpha(t_1) h_\beta(t_2) x_\alpha (d) = x_\alpha (p) h_\alpha(t_1) h_\beta(t_2)$, where $p = \frac{t_1^2 d}{t_2} \notin \mathcal{O}$.

From bi-K-invariance of D_s it follows that

$$D_s^{\Psi_E}(g) = D_s^{\Psi_E}(gw_\alpha) = \int\limits_{U(F)} D_s(ugw_\alpha)\Psi_E(u)\,du = \int\limits_{U(F)} D(u' g^{w_\alpha})\Psi_E^{w_\alpha}(u')\,du'.$$

We note that

$$g^{w_\alpha} = h_\alpha\left(\frac{t_2}{t_1}\right) h_\beta(t_2)\, x_{-\alpha}(d) = h_\alpha\left(\frac{t_2}{dt_1}\right) h_\beta(t_2)\, x_\alpha(d)\, k = x_\alpha(p')\, h_\alpha(s_1) h_\beta(s_2),$$

for some $k \in K$, $p' \in F$ and $s_1, s_2 \in F^\times$. In particular,

$$p' = \frac{d\left(\frac{t_2}{dt_1}\right)^2}{t_2} = \frac{1}{p}$$

and $|p'| < 1$. Let $|s_1| = q^{-n}$, $|s_2| = q^{-m}$ and $|p'| = q^l$.

On the other hand, denote $\widetilde{\Psi} = \Psi_E^{w\alpha}$, it holds that

$$\widetilde{\Psi}(u(r_1, r_2, r_3, r_4, r_5)) = \psi\left(r_1 + D_{(a,b,c)} r_3 - N_{(a,b,c)} r_4\right).$$

In symmetry with Lemma 1.6.2, it follows that

Lemma 4.2.10. *Let $f \in \mathcal{M}_{\Psi_E}$, then for $f(h_\alpha(s_1)h_\beta(s_2)x_\alpha(d)) = 0$ unless the following holds:*

$$\frac{s_2^2}{s_1^3} + D_{(a,b,c)} d^2 s_1 - N_{(a,b,c)} \frac{d^3 s_1^3}{s_2}, 2D_{(a,b,c)} d s_1 - N_{(a,b,c)} \frac{3d^2 s_1^3}{s_2}, D_{(a,b,c)} s_1 - N_{(a,b,c)} \frac{3ds_1^3}{s_2}, N_{(a,b,c)} \frac{s_1^3}{s_2} \in \mathcal{O}. \tag{4.14}$$

Remark 4.2.11. We note that

$$1 < |d| = \left|p' \frac{s_2}{s_1^2}\right| \Rightarrow \left|\frac{s_1^2}{s_2}\right| < |p'| < 1.$$

Remark 4.2.12. Note that Remark 4.2.9 holds in this case too.

4.2.3.1 $N_{(a,b,c)} = 0$

In this case, Equation (4.14) is equivalent to

$$\frac{s_2^2}{s_1^3} + D_{(a,b,c)} d^2 s_1, ds_1, s_1 \in \mathcal{O}$$

or rather

$$\frac{s_2^2}{s_1^3}\left(1+D_{(a,b,c)}p'^2\right), \frac{p's_2}{s_1}, s_1 \in \mathcal{O}.$$

In particular

$$\left|\frac{s_2}{s_1}\right|^2 = |s_1| \cdot \left|\frac{s_2^2}{s_1^3}\left(1+D_{(a,b,c)}p'^2\right)\right| \leq 1$$

and hence $|s_2| \leq |s_1|$.

Furthermore, $U_k(g) = \emptyset$ unless $k \geq n$ and we have $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned}
&|r_1|, |r_2|, |r_3|, |r_2r_3 + r_5|, \left|r_2^2 - r_1r_3\right| \leq q^{k+n-m} \\
&|p'r_1 - r_2|, |p'r_2 - r_3|, |p'r_3 - r_4|, \left|r_2r_4 - r_3^2 - p'r_2r_3 - p'r_5\right| \leq q^{k-n} \\
&\left|r_1r_4 - 2r_2r_3 + p'r_2^2 - p'r_1r_3 - r_5\right| \leq q^{k-n}.
\end{aligned}$$

Also note that since $|p'| < 1$ it holds that

$$|r_2| = \left|r_2\left(1+p'^2D_{(a,b,c)}\right)\right| = \left|\left(p'D_{(a,b,c)}r_3 + r_2\right) + p'D_{(a,b,c)}\left(p'r_2 - r_3\right)\right| \leq q^{k-n}$$

As before, we make a change of variables

$$\begin{aligned}
x &= r_1 + D_{(a,b,c)}r_3 \\
z &= r_2r_3 + r_5.
\end{aligned}$$

We then have $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned}
&|x| \leq q \\
&|z|, \left|r_2^2 + D_{(a,b,c)}r_3^2\right| \leq q^{k+n-m} \\
&|r_2|, |r_3|, |r_4|, \left|r_2r_4 - r_3^2 - p'z\right| \leq q^{k-n} \\
&\left|\left(D_{(a,b,c)}r_3 - x\right)\left(p'r_3 - r_4\right) + r_2\left(p'r_2 - r_3\right) - z\right| \leq q^{k-n}.
\end{aligned}$$

- $\underline{k = n}$: In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned}
&|x| \leq q \\
&|z| \leq q^{2n-m}
\end{aligned}$$

$$|r_2|, |r_3|, |r_4|, |p'z|, |x(p'r_3 - r_4) + z| \le 1.$$

We make another change of variables

$$\theta = x(p'r_3 - r_4) + z$$

and then $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|x| \le q$$
$$|z| \le q^{2n-m}$$
$$|r_2|, |r_3|, |r_4|, |\theta| \le 1.$$

Hence

$$E_n^{\widetilde{\Psi}}(g) = 0.$$

- $\underline{k > n}$: We make a change of variables

$$\theta = \left(D_{(a,b,c)}r_3 - x\right)(p'r_3 - r_4) + r_2(p'r_2 - r_3) - z$$

and note that

$$\left|r_2r_4 - r_3^2 - p'z\right| =$$
$$= \left|r_2r_4 - r_3^2 - p'\theta - -p'D_{(a,b,c)}r_3(p'r_3 - r_4) - p'r_2(p'r_2 - r_3) + p'x(p'r_3 - r_4)\right| \le q^{k-n}$$

if and only if

$$\left|r_2r_4 - r_3^2 - p'D_{(a,b,c)}r_3(p'r_3 - r_4) - p'r_2(p'r_2 - r_3)\right| \le q^{k-n}$$

since

$$|p'x(p'r_2 - r_3)| \le q^{l+1+k-n} \le q^{k-n}.$$

Thus $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$|x| \le q$$
$$|z|, \left|r_2^2 + D_{(a,b,c)}r_3^2\right| \le q^{k+n-m}$$
$$|r_2|, |r_3|, |r_4|, |\theta|, \left|r_2r_4 - r_3^2 - p'D_{(a,b,c)}r_3(p'r_3 - r_4) - p'r_2(p'r_2 - r_3)\right| \le q^{k-n}.$$

Hence

$$E_n^{\widetilde{\Psi}}(g) = 0.$$

4.2.3.2 $N_{(a,b,c)} \in \mathcal{O}^\times$

We first note that Equation (4.14) is equivalent to

$$\frac{s_2^2}{s_1^3}\left(1+D_{(a,b,c)}p'^2-N_{(a,b,c)}p'^3\right), 2D_{(a,b,c)}\frac{p's_2}{s_1}-N_{(a,b,c)}\frac{3p'^2s_2}{s_1}, D_{(a,b,c)}s_1-3N_{(a,b,c)}p's_1, N_{(a,b,c)}\frac{s_1^3}{s_2} \in \mathcal{O}.$$

Combining these two formulations of Equation (4.14), the following holds:

1. Since $d \notin \mathcal{O}$
$$D_{(a,b,c)}s_1 = \frac{1}{d}\left(2D_{(a,b,c)}ds_1 - N_{(a,b,c)}\frac{3d^2s_1^3}{s_2}\right) - D_{(a,b,c)}s_1 - N_{(a,b,c)}\frac{3ds_1^3}{s_2} \in \mathcal{O}$$
and hence $s_1 \in \mathcal{O}$.
2. As mentioned above, $\left|\frac{s_1^2}{s_2}\right| < 1$.
3. $$\left|\frac{s_2^2}{s_1^3}\right| = \left|\frac{s_2^2}{s_1^3}\left(1+D_{(a,b,c)}p'^2-N_{(a,b,c)}p'^3\right)\right| \leq 1$$
4. From the previous item, it holds that
$$\left|\frac{s_2}{s_1}\right|^2 = |s_1| \cdot \left|\frac{s_2^2}{s_1^3}\right| \leq 1$$
and in particular, $\frac{s_2}{s_1} \in \mathcal{O}$.

Furthermore, $U_k(g) = \emptyset$ unless $k \geq n$ and we have $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned}
&|r_1|, |r_2|, |r_3|, |r_2r_3+r_5|, \left|r_2^2-r_1r_3\right| \leq q^{k+n-m}\\
&|p'r_1-r_2|, |p'r_2-r_3|, |p'r_3-r_4|, \left|r_2r_4-r_3^2-p'r_2r_3-p'r_5\right| \leq q^{k-n}\\
&\left|r_1r_4-2r_2r_3+p'r_2^2-p'r_1r_3-r_5\right| \leq q^{k-n}.
\end{aligned}$$

As before, we make a change of variables

$$\begin{aligned}
x &= r_1 + D_{(a,b,c)}r_3 - N_{(a,b,c)}r_4\\
z &= r_2r_3 + r_5.
\end{aligned}$$

We then have $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned}
&|x| \leq q \\
&|z|, \left|r_2^2 + r_3\left(D_{(a,b,c)} r_3 + N_{(a,b,c)} r_4\right)\right| \leq q^{k+n-m} \\
&|r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2 - p' z\right| \leq q^{k-n} \\
&\left|\left(x - D_{(a,b,c)} r_3 + N_{(a,b,c)} r_4\right)(r_4 - p' r_3) + r_2(p' r_2 - r_3) - z\right| \leq q^{k-n}.
\end{aligned}$$

- <u>$k = n$:</u> In this case, $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned}
&|x| \leq q \\
&|z| \leq q^{2n-m} \\
&|r_2|, |r_3|, |r_4|, |p' z|, |x(r_4 - p' r_3) - z| \leq 1.
\end{aligned}$$

  We make another change of variables

$$\theta = x(r_4 - p' r_3) - z$$

  and then $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned}
&|x| \leq q \\
&|z| \leq q^{2n-m} \\
&|r_2|, |r_3|, |r_4|, |\theta| \leq 1.
\end{aligned}$$

  Hence

$$E_n^{\widetilde{\Psi}}(g) = 0.$$

- <u>$k > n$:</u> We make a change of variables

$$\theta = \left(x - D_{(a,b,c)} r_3 + N_{(a,b,c)} r_4\right)(r_4 - p' r_3) + r_2(p' r_2 - r_3) - z$$

  and note that

$$\begin{aligned}
&\left|r_2 r_4 - r_3^2 - p' z\right| \\
&= \left|r_2 r_4 - r_3^2 - p'\theta - p'(x - D_e r_3 - N_e r_4)(r_4 - p' r_3) + p' r_2(p' r_2 - r_3)\right| \leq q^{k-n}
\end{aligned}$$

if and only if

$$\left|r_2 r_4 - r_3^2 + p'\theta - p'\left(D_e r_3 + N_e r_4\right)\left(r_4 - p' r_3\right) + p' r_2\left(p' r_2 - r_3\right)\right| \leq q^{k-n}.$$

In conclusion $u(r_1, r_2, r_3, r_4, r_5) \in \widehat{U_k(g)}$ if and only if

$$\begin{aligned}
&|x| \leq q \\
&|z|, \left|r_2^2 + r_3\left(D_e r_3 + N_e r_4\right)\right| \leq q^{k+n-m} \\
&|r_2|, |r_3|, |r_4|, |\theta|, \left|r_2 r_4 - r_3^2 - p'\left(D_e r_3 + N_e r_4\right)\left(r_4 - p' r_3\right) + p' r_2\left(p' r_2 - r_3\right)\right| \leq q^{k-n}.
\end{aligned}$$

Hence

$$E_n^{\widetilde{\Psi}}(g) = 0.$$

4.3 Computing $F^*(\Psi_E, \chi, \cdot, s)$

In this subsection we prove Proposition 4.1.3. Before doing so, we prove the following result. Recall the definition of χ_s from Equation (4.6).

Lemma 4.3.1 (Theorem 4.6.5, [4])**.**

$$\int_F f_s\left(w_\beta x_\beta\left(\frac{r}{\vartheta}\right)\right)\psi(r)\, dr = \begin{cases} \frac{\mathcal{L}_F(\chi_s,-1)}{\mathcal{L}_F(\chi_s,0)}\left(|\vartheta| - \chi_s(\vartheta\,\varpi)\, q\right), & |\vartheta| \leq 1 \\ 0, & |\vartheta| > 1 \end{cases}.$$

Remark 4.3.2. Note that

$$\mathcal{L}_F(\chi_s, 0) = \mathcal{L}\left(\chi, s + \frac{5}{2}\right).$$

Proof. First we split the integral into two parts

$$\int_F f_s\left(w_\beta x_\beta\left(\frac{r}{\vartheta}\right)\right)\psi(r)\, dr = \int_{\vartheta\mathcal{O}} f_s\left(w_\beta x_\beta\left(\frac{r}{\vartheta}\right)\right)\psi(r)\, dr + \int_{F\setminus\vartheta\mathcal{O}} f_s\left(w_\beta x_\beta\left(\frac{r}{\vartheta}\right)\right)\psi(r)\, dr.$$

Considering the first integral yields

$$\int_{\vartheta\mathcal{O}} f_s\left(w_\beta x_\beta\left(\frac{r}{\vartheta}\right)\right)\psi(r)\, dr = \int_{\vartheta\mathcal{O}} \psi(r)\, dr = \begin{cases} |\vartheta|_F, & |\vartheta|_F \leq 1 \\ 0, & |\vartheta|_F > 1 \end{cases},$$

while the second integral equals

$$\begin{aligned}\int\limits_{F\setminus\vartheta\mathcal{O}} f_s\left(w_\beta x_\beta\left(\frac{r}{\vartheta}\right)\right)\psi(r)\,dr &= \int\limits_{F\setminus\vartheta\mathcal{O}} f_s\left(x_{-\beta}\left(\frac{r}{\vartheta}\right)\right)\psi(r)\,dr \\ &= \int\limits_{F\setminus\vartheta\mathcal{O}} f_s\left(x_\beta\left(\frac{\vartheta}{r}\right)h_\beta\left(\frac{\vartheta}{r}\right)\right)\psi(r)\,dr \\ &= \chi_s(\vartheta)\int\limits_{F\setminus\vartheta\mathcal{O}} \chi_s\left(\frac{1}{r}\right)\psi(r)\,dr.\end{aligned}$$

Since χ_s is $\mathcal{O}^\times$-invariant this equals

$$\int\limits_{F\setminus\vartheta\mathcal{O}} \chi_s\left(\frac{1}{r}\right)\psi(r)\,dr = \sum_{j=1-val(\vartheta)}^{\infty} \chi_s\left(\varpi^j\right)\int\limits_{|r|=q^j}\psi(r)\,dr.$$

Recall that

$$\int\limits_{|r|=q^j}\psi(r)\,dr = \begin{cases} 0, & j>1 \\ -1, & j=1 \\ q^j\left(1-q^{-1}\right), & j\le 0 \end{cases}.$$

Inserting this into the previous expression yields

$$\int\limits_{F\setminus\vartheta\mathcal{O}} \chi_s\left(\frac{1}{r}\right)\psi(r)\,dr = \begin{cases} 0, & |\vartheta|_F>1 \\ -\chi_s(\varpi), & |\vartheta|_F=1 \\ \left(1-q^{-1}\right)\left(\frac{1-\chi_s\left(\frac{1}{\vartheta}\right)|\vartheta|_F}{1-\chi_s(\varpi^{-1})|\varpi|_F}\right)-\chi_s(\varpi), & |\vartheta|_F<1 \end{cases}.$$

Combining all of the above yields the lemma. □

Remark 4.3.3. For the convenience of displaying the following computation we introduce another notation for this section. For a triple $r=(r_1,r_3,r_4)\in F_{\alpha_1}\times F_{\alpha_3}\times F_{\alpha_4}$ we write

$$\begin{aligned}x_\alpha(r) &= x_{\alpha_1}(r_1)\,x_{\alpha_3}(r_3)\,x_{\alpha_4}(r_4)\\ x_{-\alpha}(r) &= x_{-\alpha_1}(r_1)\,x_{-\alpha_3}(r_3)\,x_{-\alpha_4}(r_4).\end{aligned}$$

We do not apply this notation to the other roots of G.

Note that this notation is not in conflict with the definition of x_α for the root α in G, namely for $d \in F$ let $d' = (d, d, d)$ and then $x_\alpha(d) = x_\alpha(d')$ is in $G(F) \subseteq H_E(F)$.

We also denote $e = (a, b, c)$, where (a, b, c) are as in **(CT)**.

Also, for $x \in F_{\alpha_i}$ recall that for $|x|_{F_{\alpha_i}} > 1$ it holds that

$$x_{\alpha_i}(-x)\, h_{\alpha_i}(x)\, x_{-\alpha_i}(x) \in H_E(\mathcal{O}). \tag{4.15}$$

Proof of Proposition 4.1.3. We recall that F_{α_i} is the field of definition of the root α_i in H_E as described in Section 1.7. Conjugating w_α to the right and using left (P_E, χ_s)-invariance and right $H_E(\mathcal{O})$-invariance we have

$$\begin{aligned} F^*(\Psi_E, \chi, g, s) &= \int\limits_F f_s\left(w_\beta w_\alpha x_\alpha(e)\, x_{3\alpha+\beta}(r)\, h_\alpha(t_1)\, h_\beta(t_2)\, x_\alpha(d)\right) \psi(r)\, dr \\ &= \chi_s\left(\frac{t_2^2}{t_1^3}\right) \int\limits_F f_s\left(w_\beta x_\beta\left(\frac{t_2}{t_1^3} r\right) x_{-\alpha}\left(\frac{t_2}{t_1^2} e + d\right)\right) \psi(r)\, dr. \end{aligned}$$

We continue to consider this integral case-by-case.

Case I: Assume that $\left|\frac{t_2}{t_1^2} a + d\right|_{F_{\alpha_1}}, \left|\frac{t_2}{t_1^2} b + d\right|_{F_{\alpha_3}}, \left|\frac{t_2}{t_1^2} c + d\right|_{F_{\alpha_4}} \le 1$.

In this case we have

$$F^*(\Psi_E, \chi, g, s) = \chi_s\left(\frac{t_2^2}{t_1^3}\right) \int\limits_F f_s\left(w_\beta x_\beta\left(\frac{t_2}{t_1^3} r\right)\right) \psi(r)\, dr.$$

Applying Lemma 4.3.1 with $\vartheta = \frac{t_1^3}{t_2}$ yields the assertion.

Case II: The case where $\left|\frac{t_2}{t_1^2} a + d\right|_{F_{\alpha_1}} > 1$, $\left|\frac{t_2}{t_1^2} b + d\right|_{F_{\alpha_3}} \le 1$ and $\left|\frac{t_2}{t_1^2} c + d\right|_{F_{\alpha_4}} \le 1$ (or the cases equivalent to this) cannot happen under **(CT)** for a non-split E.

Case III: Assume $\left|\frac{t_2}{t_1^2} a + d\right|_{F_{\alpha_1}} \le 1$, $\left|\frac{t_2}{t_1^2} b + d\right|_{F_{\alpha_3}} > 1$ and $\left|\frac{t_2}{t_1^2} c + d\right|_{F_{\alpha_4}} > 1$ (the cases equivalent to this follow similarly). Applying Equation (4.15) and conjugating the elements yields

$$F^*(\Psi_E, \chi, g, s) = \chi_s\left(\frac{t_2^2}{t_1^3}\right) \int\limits_F f_s\left(w_\beta x_\beta\left(\frac{t_2}{t_1^3} r\right) x_{-\alpha_1}\left(\frac{t_2}{t_1^2} b + d\right) x_{-\alpha_3}\left(\frac{t_2}{t_1^2} c + d\right)\right) \psi(r)\, dr$$

$$= \chi_s\left(\frac{t_2^2}{t_1^3}\left(\frac{1}{\frac{t_2}{t_1^2}b+d}\cdot\frac{1}{\frac{t_2}{t_1^2}c+d}\right)\right)\int\limits_F f_s\left(w_\beta x_\beta\left(\frac{t_2}{t_1^3}r\left(\frac{1}{\frac{t_2}{t_1^2}b+d}\cdot\frac{1}{\frac{t_2}{t_1^2}c+d}\right)\right)\right)\psi(r)\,dr$$

Applying Lemma 4.3.1 with $\vartheta = \frac{t_1^3}{t_2}\left(\frac{t_2}{t_1^2}b+d\right)\left(\frac{t_2}{t_1^2}c+d\right)$ yields the assertion.

<u>Case IV:</u> Assume $\left|\frac{t_2}{t_1^2}a+d\right|_{F_{\alpha_1}} > 1$, $\left|\frac{t_2}{t_1^2}b+d\right|_{F_{\alpha_3}} > 1$ and $\left|\frac{t_2}{t_1^2}c+d\right|_{F_{\alpha_4}} > 1$. We then have

$$F^*(\Psi_E,\chi,g,s) = \chi_s\left(\frac{t_2^2}{t_1^3}\right)\int\limits_F f_s\left(w_\beta x_\beta\left(\frac{t_2}{t_1^3}r\right)x_{-\alpha}\left(\frac{t_2}{t_1^2}e+d\right)\right)\psi(r)\,dr$$

$$= \chi_s\left(\frac{\frac{t_2^2}{t_1^3}}{\mathrm{Nm}\left(\frac{1}{\frac{t_2}{t_1^2}e+d}\right)}\right)\int\limits_F f_s\left(w_\beta x_\beta\left(\frac{\frac{t_2}{t_1^3}r}{\mathrm{Nm}\left(\frac{t_2}{t_1^2}e+d\right)}\right)x_\alpha\left(\frac{t_2}{t_1^2}e+d\right)\right)\psi(r)\,dr.$$

Applying Lemma 4.3.1 with $\vartheta = \frac{t_1^3}{t_2}\left(\frac{t_2}{t_1^2}a+d\right)\left(\frac{t_2}{t_1^2}b+d\right)\left(\frac{t_2}{t_1^2}c+d\right)$ yields the assertion.

□

As explained in Section 4.1, in what follows we may assume that $\chi = \mathbb{1}$. The following, more explicit, formula for $F^*(\Psi_E,\mathbb{1},g,s)$ is used for the computation of the convolution $F^*(\Psi_E,\mathbb{1},\cdot,s) * P_s(\cdot)$.

Corollary 4.3.4. *Let $g = h_\alpha(t_1)\,h_\beta(t_2)\,x_\alpha(d)$. Further assume that g satisfies Equation (1.7). Then:*

- *<u>$E = F\times K$:</u> For $d = 0$ it holds that*

$$F(\Psi_E,\mathbb{1},g,s) = \begin{cases} \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}|t_2|_F^{2s+4}|t_1|_F^{-3s-\frac{9}{2}}\left(1-\left|\frac{t_1^3}{t_2}\right|_F^{s+\frac{3}{2}}q^{-s-\frac{3}{2}}\right), & \left|\frac{t_2}{t_1^2}\right|_F \le 1\\ \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}|t_2|_F|t_1|_F^{s+\frac{3}{2}}\left(1-\left|\frac{t_2}{t_1}\right|_F^{s+\frac{3}{2}}q^{-s-\frac{3}{2}}\right), & \left|\frac{t_2}{t_1^2}\right|_F > 1\end{cases}$$

and for $d\notin\mathcal{O}$, it holds that

$$F(\Psi_E,\mathbb{1},g,s)=\begin{cases}\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}|t_2|_F\left|\frac{d}{t_1}\right|_F^{-s-\frac{3}{2}}\left(1-\left|\frac{dt_2}{t_1}\right|_F^{s+\frac{3}{2}}q^{-s-\frac{3}{2}}\right), & \left|\frac{dt_1^2}{t_2}\right|_F\le 1\&\left|\frac{dt_2}{t_1}\right|_F\le 1\\ \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}|t_2|_F^{2s+4}|dt_1|_F^{-3s-\frac{9}{2}}\left(1-\left|\frac{d^3t_1^3}{t_2}\right|_F^{s+\frac{3}{2}}q^{-s-\frac{3}{2}}\right), & \left|\frac{dt_1^2}{t_2}\right|_F>1\&\left|\frac{d^3t_1^3}{t_2}\right|_F\le 1\\ 0, & \left|\frac{dt_1^2}{t_2}\right|_F\le 1\&\left|\frac{dt_2}{t_1}\right|_F>1\\ 0, & \left|\frac{dt_1^2}{t_2}\right|_F>1\&\left|\frac{d^3t_1^3}{t_2}\right|_F>1\end{cases}.$$

- *<u>E a field</u>: For $d=0$ it holds that*

$$F(\Psi_E,\mathbb{1},g,s)=\begin{cases}\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}|t_2|_F^{2s+4}|t_1|_F^{-3s-\frac{9}{2}}\left(1-\left|\frac{t_1^3}{t_2}\right|_F^{s+\frac{3}{2}}q^{-s-\frac{3}{2}}\right), & \left|\frac{t_2}{t_1^2}\right|_F\le 1\\ \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}|t_2|_F^{-s-\frac{1}{2}}|t_1|_F^{3s+\frac{9}{2}}\left(1-\left|\frac{t_2^2}{t_1^3}\right|_F^{s+\frac{3}{2}}q^{-s-\frac{3}{2}}\right), & \left|\frac{t_2}{t_1^2}\right|_F>1\end{cases}$$

and for $d\notin\mathcal{O}$, it holds that

$$F(\Psi_E,\mathbb{1},g,s)=\begin{cases}\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}|t_2|_F^{-s-\frac{1}{2}}|t_1|_F^{3s+\frac{9}{2}}\left(1-\left|\frac{t_2^2}{t_1^3}\right|_F^{s+\frac{3}{2}}q^{-s-\frac{3}{2}}\right), & \left|\frac{dt_1^2}{t_2}\right|_F\le 1\&\left|\frac{t_2^2}{t_1^3}\right|_F\le 1\\ \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}|t_2|_F^{2s+4}|dt_1|_F^{-3s-\frac{9}{2}}\left(1-\left|\frac{d^3t_1^3}{t_2}\right|_F^{s+\frac{3}{2}}q^{-s-\frac{3}{2}}\right), & \left|\frac{dt_1^2}{t_2}\right|_F>1\&\left|\frac{d^3t_1^3}{t_2}\right|_F\le 1\\ 0, & \left|\frac{dt_1^2}{t_2}\right|_F\le 1\&\left|\frac{t_2^2}{t_1^3}\right|_F>1\\ 0, & \left|\frac{dt_1^2}{t_2}\right|_F>1\&\left|\frac{d^3t_1^3}{t_2}\right|_F>1\end{cases}.$$

Proof. We prove the corollary for the case $E=F\times K$; the case where E is a field is proven similarly. Assuming $d=0$, it holds that

$$a_1=\begin{cases}1, & \left|\frac{t_2}{t_1^2}\right|\le 1\\ \frac{t_2}{t_1^2}a, & \left|\frac{t_2}{t_1^2}\right|>1\end{cases},\quad a_2=\begin{cases}1, & \left|\frac{t_2}{t_1^2}\right|\le 1\\ \frac{t_2}{t_1^2}b, & \left|\frac{t_2}{t_1^2}\right|>1\end{cases},\quad a_3=\begin{cases}1, & \left|\frac{t_2}{t_1^2}\right|\le 1\\ \frac{t_2}{t_1^2}c, & \left|\frac{t_2}{t_1^2}\right|>1\end{cases}.$$

Hence

$$\vartheta=\begin{cases}\frac{t_1^3}{t_2}, & \left|\frac{t_2}{t_1^2}\right|\le 1\\ \frac{t_2}{t_1}D_{a,b,c}, & \left|\frac{t_2}{t_1^2}\right|>1\end{cases},\qquad |\vartheta|=\begin{cases}\left|\frac{t_1^3}{t_2}\right|, & \left|\frac{t_2}{t_1^2}\right|\le 1\\ \left|\frac{t_2}{t_1}\right|, & \left|\frac{t_2}{t_1^2}\right|>1\end{cases}.$$

This yields the assertion.

Assuming that $d \notin \mathcal{O}$, in this case it holds that

$$a_1 = d, \quad a_2 = \frac{t_2}{t_1^2}\theta + d, \quad a_3 = \frac{t_2}{t_1^2}\theta^\sigma + d.$$

Hence

$$\vartheta = \begin{cases} \frac{dt_2}{t_1}, & \left|\frac{dt_1^2}{t_2}\right| \leq 1 \\ \frac{dt_2}{t_1}\left(\left(\frac{dt_1^2}{t_2}\right)^2 + D_{a,b,c}\right), & \left|\frac{dt_1^2}{t_2}\right| > 1 \end{cases}, \qquad |\vartheta| = \begin{cases} \left|\frac{dt_2}{t_1}\right|, & \left|\frac{dt_1^2}{t_2}\right| \leq 1 \\ \left|\frac{d^3t_1^3}{t_2}\right|, & \left|\frac{dt_1^2}{t_2}\right| > 1 \end{cases}.$$

This yields the assertion. □

4.4 Computing $F^*(\Psi_E, \mathbb{1}, \cdot, s) * P_s(\cdot)$

As mentioned in Section 4.1, we consider only the case where E is a non-split Galois étale cubic algebra over F. In this section we explain the key ideas needed to compute the convolution $F^*(\Psi_E, \mathbb{1}, \cdot, s) * P_s(\cdot)(g)$. The computation of the convolution for any g is performed in Appendix A.

We recall from Section 4.1 that

$$P_s = \frac{R_0\left(q^{-s-\frac{1}{2}}\right)A_0 - R_1\left(q^{-s-\frac{1}{2}}\right)A_1}{\xi\left(s+\frac{3}{2}\right)\xi\left(s+\frac{7}{2}\right)\xi\left(s+\frac{1}{2}\right)}.$$

As $A_0 = \mathbb{1}_K$ and $A_1 = q^{-3}\left(\mathbb{1}_K + \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)$, in order to compute $F^*(\Psi_E, \mathbb{1}, \cdot, s) * P_s(\cdot)$ we only need to compute $F^*(\Psi_E, \mathbb{1}, \cdot, s) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}$. For any $f \in \mathcal{M}_{\Psi_E}$ it holds that

$$f * \mathbb{1}_{K\omega_2^\vee(\varpi)K}(g) = \int\limits_{K\omega_2^\vee(\varpi)K} f\left(gh^{-1}\right)dh = \sum_{\gamma\in K\backslash\left(K\omega_2^\vee(\varpi)K\right)} f\left(g\gamma^{-1}\right). \tag{4.16}$$

Let $g = h_\alpha(t_1)\, h_\beta(t_2)\, x_\alpha(d)$ and assume that if $d \in \mathcal{O}$ then $d = 0$ (we may do this by right K-invariance). Fix the list of representatives of $K\backslash\left(K\omega_2^\vee(\varpi)K\right)$ obtained in [24, Appendix A.3] from [14, Propositions 13.3 and 14.2]. The representatives are listed in the following table as $K\omega_2^\vee(\varpi)K = \coprod_i b_i' K$. We list the set of representatives b_i' in the following table. The representatives are separated here into conjugacy classes under $M(\mathcal{O})$. Let Y denote a set of representatives in $\mathcal{O}$ of $\mathcal{O}/\varpi\,\mathcal{O}$ and let Z denote a set of representatives in $\mathcal{O}$ of $\mathcal{O}/\varpi^2\,\mathcal{O}$.

Class $mod\ M(\mathcal{O})$	# cosets	Representatives
$h_{-\omega_2}(\varpi)$	1	$h_{-\omega_2}(\varpi)$
$h_{-\beta}(\varpi)$	q^6	$u(r_1, r_2, r_3, r_4, r_5)\, h_{\omega_2}(\varpi)$ $r_1, r_2, r_3, r_4 \in Y,\ r_5 \in Z$
$h_\alpha(\varpi)\, h_\beta(\varpi)$	$q(q+1)$	$u(r_1, 0, 0, 0, 0)\, h_{-\alpha}(\varpi)\, h_{-\beta}(\varpi)$ $r_1 \in Y$
		$u(0, 0, 0, r_4, 0)\, x_\alpha(z)\, h_{-\beta}(\varpi)$ $r_4, z \in Y$
$h_{\omega_2}(\varpi)$	$q^4(q+1)$	$u(r_1, r_2, 0, 0, r_5)\, h_\beta(\varpi)$ $r_2, r_5 \in Y,\ r_1 \in Z$
		$u(0, 0, r_3, r_4, r_5)\, x_\alpha(z)\, h_\alpha(\varpi)\, h_\beta(\varpi)$ $r_3, r_5, z \in Y,\ r_4 \in Z$
1	$q^3 - 1$	$u\left(0, 0, 0, 0, \frac{r_5}{\varpi}\right)$ $r_5 \in Y,\ r_5 \not\equiv 0$
		$u\left(\frac{r_1}{\varpi}, 0, 0, 0, \frac{r_5}{\varpi}\right)$ $r_1, r_5 \in Y,\ r_1 \not\equiv 0$
		$u\left(\frac{y^3 r_1}{\varpi}, \frac{y^2 r_1}{\varpi}, \frac{y r_1}{\varpi}, \frac{r_1}{\varpi}, \frac{r_5}{\varpi}\right)$ $r_1, r_5, y \in Y,\ r_1 \not\equiv 0$

Let us introduce a few notations for the rest of this section. For $g \in G(F)$ let $F_s(g) = F(\Psi_E, \mathbb{1}, g, s)$ We also write (t_1, t_2) for $h_\alpha(t_1)\, h_\beta(t_2)$.

Let $g = (t_1, t_2)\, x_\alpha(d)$ and assume that:

1. If $d \in \mathcal{O}$ then $d = 0$.

2. t_1, t_2 and d are as in Equation (1.7).

Inserting the representatives b_i into Equation (4.16) we obtain

$$\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= F_s\left(\left(\frac{t_1}{\varpi}, \frac{t_2}{\varpi^2}\right) x_\alpha\left(d\right)\right) + q^6 F_s\left(\left(\varpi\, t_1, \varpi^2\, t_2\right) x_\alpha\left(d\right)\right) \\
&+ qF_s\left(\left(\frac{t_1}{\varpi}, \frac{t_2}{\varpi}\right) x_\alpha\left(\varpi\, d\right)\right) + q \sum_{s\in\mathcal{O}/(\varpi)} F_s\left(\left(t_1, \frac{t_2}{\varpi}\right) x_\alpha\left(\frac{d-s}{\varpi}\right)\right) \\
&+ q^4 F_s\left(\left(t_1, \varpi\, t_2\right) x_\alpha\left(\varpi\, d\right)\right) + q^4 \sum_{s\in\mathcal{O}/(\varpi)} F_s\left(\left(\varpi\, t_1, \varpi\, t_2\right) x_\alpha\left(\frac{d-s}{\varpi}\right)\right) \\
&+ \left(GS\left(\Psi_E, g\right)\right) F_s\left(\left(t_1, t_2\right) x_\alpha\left(d\right)\right),
\end{aligned} \tag{4.17}$$

where GS denotes the following *Gaussian sum*:

$$\begin{aligned}
GS\left(\Psi_E, g\right) = (q-1) + q \sum_{\substack{r\in\mathcal{O}/(\varpi) \\ r\not\equiv 0}} \Psi_E\left(g x_\beta\left(\frac{r}{\varpi}\right) g^{-1}\right) \\
+ q \sum_{\substack{r,y\in\mathcal{O}/(\varpi) \\ r\not\equiv 0}} \Psi_E\left(gu\left(\frac{y^3 r}{\varpi}, \frac{y^2 r}{\varpi}, \frac{yr}{\varpi}, \frac{r}{\varpi}\right)^{-1} g^{-1}\right).
\end{aligned} \tag{4.18}$$

Lemma 4.4.1. *Assuming that $g = h_\alpha\left(t_1\right) h_\beta\left(t_2\right) x_\alpha\left(d\right)$ is as in Equation (1.7), it then holds that*

- *$E = F \times K$: For $d = 0$ it holds that*

$$GS\left(\Psi_E, g\right) = \begin{cases} q^3 - 1, & \left|\frac{t_2}{t_1^2}\right| \leq 1 \;\&\; \left|\frac{t_1^3}{t_2}\right| < 1 \;\; or \;\; \left|\frac{t_2}{t_1^2}\right| > 1 \;\&\; \left|\frac{t_2}{t_1}\right| < 1 \\ -1, & \left|\frac{t_2}{t_1^2}\right| \leq 1 \;\&\; \left|\frac{t_1^3}{t_2}\right| = 1 \\ q^2 - 1. & \left|\frac{t_2}{t_1^2}\right| > 1 \;\&\; \left|\frac{t_2}{t_1}\right| = 1 \end{cases}$$

and for $d \notin \mathcal{O}$, it holds that

$$GS\left(\Psi_E, g\right) = \begin{cases} q^3 - 1, & \left|\frac{dt_1^2}{t_2}\right| \leq 1 \;\&\; \left|\frac{dt_2}{t_1}\right| < 1 \;\; or \;\; \left|\frac{dt_1^2}{t_2}\right| > 1 \;\&\; \left|\frac{d^3 t_1^3}{t_2}\right| < 1 \\ -1, & \left|\frac{dt_1^2}{t_2}\right| \leq 1 \;\&\; \left|\frac{dt_2}{t_1}\right| = 1 \;\; or \;\; \left|\frac{dt_1^2}{t_2}\right| > \; 1\& \; \left|\frac{d^3 t_1^3}{t_2}\right| = 1 \end{cases}.$$

- *E a field: For $d = 0$ it holds that*

$$GS(\Psi_E, g) = \begin{cases} q^3 - 1, & \left|\frac{t_2}{t_1^2}\right| \leq 1 \ \& \ \left|\frac{t_1^3}{t_2}\right| < 1 \ or \ \left|\frac{t_2}{t_1^2}\right| > 1 \ \& \ \left|\frac{t_2^2}{t_1^3}\right| < 1 \\ -1, & \left|\frac{t_2}{t_1^2}\right| < 1 \ \& \ \left|\frac{t_1^3}{t_2}\right| = 1 \ or \ \left|\frac{t_2}{t_1^2}\right| > 1 \ \& \ \left|\frac{t_2^2}{t_1^3}\right| = 1 \\ -q^2 - 1, & \left|\frac{t_2}{t_1^2}\right| = 1 \ \& \ \left|\frac{t_1^3}{t_2}\right| = 1 \end{cases}$$

and for $d \notin \mathcal{O}$, it holds that

$$GS(\Psi_E, g) = \begin{cases} q^3 - 1, & \left|\frac{dt_1^2}{t_2}\right| \leq 1 \ \& \ \left|\frac{t_2^2}{t_1^3}\right| < 1 \ or \ \left|\frac{dt_1^2}{t_2}\right| > 1 \ \& \ \left|\frac{d^3t_1^3}{t_2}\right| < 1 \\ -1, & \left|\frac{dt_1^2}{t_2}\right| \leq 1 \ \& \ \left|\frac{t_2^2}{t_1^3}\right| = 1 \ or \ \left|\frac{dt_1^2}{t_2}\right| > 1 \ \& \ \left|\frac{d^3t_1^3}{t_2}\right| = 1 \end{cases}.$$

Proof. For the convenience of the reader, we start by proving this lemma for the case where $d = 0$ and then proceed to the case $d \notin \mathcal{O}$. The proof relies on the fact that

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \psi(\varpi^n x) = \begin{cases} 0, & n < -1 \\ -1, & n = -1 \\ q - 1, & n \geq 0 \end{cases}.$$

We first deal with the first sum in Equation (4.18)

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \Psi_E\left(g x_\beta\left(\frac{r}{\varpi}\right) g^{-1}\right) = \sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \psi\left(N_{(a,b,c)} \frac{t_2^2}{t_1^3} \frac{r}{\varpi}\right).$$

- $\underline{E = F \times K}$: In this case $N_{(a,b,c)} = 0$ due to **(CT)** and hence

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \Psi_E\left(g x_\beta\left(\frac{r}{\varpi}\right) g^{-1}\right) = q - 1.$$

- $\underline{E \text{ is a field}}$: In this case $N_{(a,b,c)} \in \mathcal{O}^\times$ and hence

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \Psi_E\left(g x_\beta\left(\frac{r}{\varpi}\right) g^{-1}\right) = \begin{cases} q - 1, & \left|\frac{t_2^2}{t_1^3}\right| < 1 \\ -1, & \left|\frac{t_2^2}{t_1^3}\right| = 1 \end{cases}.$$

Here we assume that $\left|\frac{t_2^2}{t_1^3}\right| \leq 1$ in accordance with Remark 4.2.6.

We now compute the second sum in Equation (4.18)

$$\sum_{\substack{r,y\in\mathcal{O}/(\varpi)\\ r\not\equiv 0}} \Psi_E\left(gu\left(\frac{y^3r}{\varpi},\frac{y^2r}{\varpi},\frac{yr}{\varpi},\frac{r}{\varpi}\right)^{-1}g^{-1}\right)$$
$$=\sum_{\substack{r,y\in\mathcal{O}/(\varpi)\\ r\not\equiv 0}} \psi\left(\frac{r}{\varpi}\left(N_{(a,b,c)}\frac{t_2^2}{t_1^3}y^3+D_{(a,b,c)}\frac{t_2}{t_1}y^2+\frac{t_1^3}{t_2}\right)\right)$$
$$=\sum_{\substack{r\in\mathcal{O}/(\varpi)\\ r\not\equiv 0}} \psi\left(\frac{r}{\varpi}\frac{t_1^3}{t_2}\right)+\sum_{\substack{y\in\mathcal{O}/(\varpi)\\ y\not\equiv 0}}\sum_{\substack{r\in\mathcal{O}/(\varpi)\\ r\not\equiv 0}} \psi\left(\frac{r}{\varpi}\left(N_{(a,b,c)}\frac{t_2^2}{t_1^3}y^3+D_{(a,b,c)}\frac{t_2}{t_1}y^2+\frac{t_1^3}{t_2}\right)\right).$$

We first have

$$\sum_{\substack{r\in\mathcal{O}/(\varpi)\\ r\not\equiv 0}} \psi\left(\frac{r}{\varpi}\frac{t_1^3}{t_2}\right)=\begin{cases} q-1, & \left|\frac{t_1^3}{t_2}\right|<1\\ -1, & \left|\frac{t_1^3}{t_2}\right|=1\end{cases}.$$

Here we assume again that $\left|\frac{t_2^2}{t_1^3}\right|\leq 1$ in accordance with Remark 4.2.6.

- <u>$E=F\times K$:</u> In this case $N_{(a,b,c)}=0$. For $y\not\equiv 0$ it then holds that

$$\left|\left(N_{(a,b,c)}\frac{t_2^2}{t_1^3}y^3+D_{(a,b,c)}\frac{t_2}{t_1}y^2+\frac{t_1^3}{t_2}\right)\right|=\left|\frac{t_1^3}{t_2}\left(D_{(a,b,c)}\left(\frac{t_2}{t_1^2}y\right)^2+1\right)\right|=\begin{cases}\left|\frac{t_1^3}{t_2}\right|, & \left|\frac{t_2}{t_1^2}\right|\leq 1\\ \left|\frac{t_2}{t_1}\right|, & \left|\frac{t_2}{t_1^2}\right|>1\end{cases}.$$

  This follows from the fact that K/F is unramified and hence $D_{(a,b,c)}\left(\frac{t_2}{t_1}y\right)^2+1\in\mathcal{O}^\times$. It follows that

$$\sum_{\substack{r\in\mathcal{O}/(\varpi)\\ r\not\equiv 0}} \psi\left(\frac{r}{\varpi}\left(N_{(a,b,c)}\frac{t_2^2}{t_1^3}y^3+D_{(a,b,c)}\frac{t_2}{t_1}y^2+\frac{t_1^3}{t_2}\right)\right)=\begin{cases} q-1, & \left|\frac{t_2}{t_1^2}\right|\leq 1\&\left|\frac{t_1^3}{t_2}\right|<1\\ & \left|\frac{t_2}{t_1^2}\right|>1\&\left|\frac{t_2}{t_1}\right|<1\\ -1, & \left|\frac{t_2}{t_1^2}\right|\leq 1\&\left|\frac{t_1^3}{t_2}\right|=1\\ & \left|\frac{t_2}{t_1^2}\right|>1\&\left|\frac{t_2}{t_1}\right|=1\end{cases}.$$

- <u>E is a field:</u> We further assume that $\left|\frac{t_1^3}{t_2}\right|\leq\left|\frac{t_2^2}{t_1^3}\right|\leq 1$ in accordance with Remark 4.2.6. In this case $N_{a,b,c} \in \mathcal{O}$. For $y \not\equiv 0$ it then holds that

$$\begin{aligned}
&\left| N_{(a,b,c)} \frac{t_2^2}{t_1^3} y^3 + D_{(a,b,c)} \frac{t_2}{t_1} y^2 + \frac{t_1^3}{t_2} \right| \\
&= \left| \frac{t_1^3}{t_2} \left(N_{(a,b,c)} \left(\frac{t_2}{t_1^2} y \right)^3 + D_{(a,b,c)} \left(\frac{t_2}{t_1^2} y \right)^2 + 1 \right) \right| = \begin{cases} \left|\frac{t_1^3}{t_2}\right|, & \left|\frac{t_2}{t_1^2}\right| \le 1 \\ \left|\frac{t_2^2}{t_1^3}\right|, & \left|\frac{t_2}{t_1^2}\right| > 1 \end{cases}.
\end{aligned}$$

This follows from the fact that E/F is unramified and hence $N_{(a,b,c)} \left(\frac{t_2}{t_1^2} y\right)^3 + D_{(a,b,c)} \left(\frac{t_2}{t_1} y\right)^2 + 1 \in \mathcal{O}^\times$. It follows that

$$\sum_{\substack{r \in A \\ r \not\equiv 0}} \psi \left(\frac{t_1^3}{t_2} \frac{r}{\varpi} \left(N_{(a,b,c)} \left(\frac{t_2}{t_1^2} y \right)^3 + D_{(a,b,c)} \left(\frac{t_2}{t_1^2} y \right)^2 + 1 \right) \right) = \begin{cases} q-1, & \left|\frac{t_2}{t_1^2}\right| \le 1 \& \left|\frac{t_1^3}{t_2}\right| < 1 \\ & \left|\frac{t_2}{t_1^2}\right| > 1 \& \left|\frac{t_2^2}{t_1^3}\right| < 1 \\ -1, & \left|\frac{t_2}{t_1^2}\right| \le 1 \& \left|\frac{t_1^3}{t_2}\right| = 1 \\ & \left|\frac{t_2}{t_1^2}\right| > 1 \& \left|\frac{t_2^2}{t_1^3}\right| = 1 \end{cases}.$$

Combining all of the above yields the assertion for $d = 0$.

Assume $d \notin \mathcal{O}$. Going back to Equation (4.18), the first sum there equals

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \Psi_E \left(g x_\beta \left(\frac{r}{\varpi} \right) g^{-1} \right) = \sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \psi \left(\left(\frac{d^3 t_1^3}{t_2} + D_{(a,b,c)} \frac{d t_2}{t_1} - N_{(a,b,c)} \frac{t_2^2}{t_1^3} \right) \frac{r}{\varpi} \right).$$

- <u>$E = F \times K$:</u> In this case $N_{(a,b,c)} = 0$ due to **(CT)** and hence

$$\left| \frac{d^3 t_1^3}{t_2} + D_{(a,b,c)} \frac{d t_2}{t_1} - N_{(a,b,c)} \frac{t_2^2}{t_1^3} \right| = \left| \left(\left(\frac{d t_1^2}{t_2} \right)^2 + D_{(a,b,c)} \right) \frac{d t_2}{t_1} \right| = \begin{cases} \left|\frac{d t_2}{t_1}\right|, & \left|\frac{d t_1^2}{t_2}\right| \le 1 \\ \left|\frac{d^3 t_1^3}{t_2}\right|, & \left|\frac{d t_1^2}{t_2}\right| > 1 \end{cases}.$$

  Therefore

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \not\equiv 0}} \Psi_E \left(g x_\beta \left(\frac{r}{\varpi} \right) g^{-1} \right) = \begin{cases} -1, & \left|\frac{d t_1^2}{t_2}\right| \le 1 \& \left|\frac{d t_2}{t_1}\right| = 1 \\ & \left|\frac{d t_1^2}{t_2}\right| > 1 \& \left|\frac{d^3 t_1^3}{t_2}\right| = 1 \\ q-1, & \left|\frac{d t_1^2}{t_2}\right| \le 1 \& \left|\frac{d t_2}{t_1}\right| < 1 \\ & \left|\frac{d t_1^2}{t_2}\right| > 1 \& \left|\frac{d^3 t_1^3}{t_2}\right| < 1 \end{cases}.$$

- $\underline{E \text{ is a field:}}$ In this case $N_{(a,b,c)} \in \mathcal{O}^\times$ and hence

$$\left|\frac{d^3t_1^3}{t_2} + D_{(a,b,c)}\frac{dt_2}{t_1} - N_{(a,b,c)}\frac{t_2^2}{t_1^3}\right| = \left|\left(\left(\frac{dt_1^2}{t_2}\right)^3 + D_{(a,b,c)}\frac{dt_1^2}{t_1} - N_{(a,b,c)}\right)\frac{t_2^2}{t_1^3}\right| = \begin{cases} \left|\frac{t_2^2}{t_1^3}\right|, & \left|\frac{dt_1^2}{t_2}\right| \le 1 \\ \left|\frac{d^3t_1^3}{t_2^2}\right|, & \left|\frac{dt_1^2}{t_2}\right| > 1 \end{cases}.$$

Therefore

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \neq 0}} \Psi_E\left(gx_\beta\left(\frac{r}{\varpi}\right)g^{-1}\right) = \begin{cases} -1, & \left|\frac{dt_1^2}{t_2}\right| \le 1 \& \left|\frac{t_2^2}{t_1^3}\right| = 1 \\ & \left|\frac{dt_1^2}{t_2}\right| > 1 \& \left|\frac{d^3t_1^3}{t_2^2}\right| = 1 \\ q-1, & \left|\frac{dt_1^2}{t_2}\right| \le 1 \& \left|\frac{t_2^2}{t_1^3}\right| < 1 \\ & \left|\frac{dt_1^2}{t_2}\right| > 1 \& \left|\frac{d^3t_1^3}{t_2^2}\right| < 1 \end{cases}.$$

The second sum in Equation (4.18) equals

$$\sum_{\substack{r,y \in \mathcal{O}/(\varpi) \\ r \neq 0}} \Psi_E\left(gu\left(\frac{y^3r}{\varpi}, \frac{y^2r}{\varpi}, \frac{yr}{\varpi}, \frac{r}{\varpi}\right)^{-1} g^{-1}\right)$$
$$= \sum_{\substack{r,y \in \mathcal{O}/(\varpi) \\ r \neq 0}} \psi\left(\left(\frac{t_1^3}{t_2}\left((dy)^3 + 3(dy)^2 + 3dy + 1\right) + D_{(a,b,c)}\frac{t_2}{t_1}\left(dy^3 + y^2\right) - N_{(a,b,c)}\frac{t_2^2}{t_1^3}y^3\right)\frac{r}{\varpi}\right)$$
$$= \sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \neq 0}} \psi\left(\frac{t_1^3}{t_2}\frac{r}{\varpi}\right) + \sum_{\substack{r,y \in \mathcal{O}/(\varpi) \\ r,y \neq 0}} \psi\left(\left(\frac{t_1^3}{t_2}\left((dy)^3 + 3(dy)^2 + 3dy + 1\right) + D_{(a,b,c)}\frac{t_2}{t_1}\left(dy^3 + y^2\right) - N_{(a,b,c)}\frac{t_2^2}{t_1^3}y^3\right)\frac{r}{\varpi}\right).$$

It follows from Lemma 1.6.2 that $\left|\frac{t_1^3}{t_2}\right| < 1$ and hence

$$\sum_{\substack{r \in \mathcal{O}/(\varpi) \\ r \neq 0}} \psi\left(\frac{r}{\varpi}\frac{t_1^3}{t_2}\right) = q - 1.$$

- $\underline{E = F \times K}$: In this case Lemma 1.6.2 implies further that

$$\left|\frac{dt_1^3}{t_2}\right|, \left|\frac{d^2t_1^3}{t_2} + D_{(a,b,c)}\frac{t_2}{t_1}\right| < 1.$$

For $y \not\equiv 0$ it holds that

$$\left|\frac{t_1^3}{t_2}\left(d^3y^3+3d^2y^2+3dy+1\right)+D_{(a,b,c)}\frac{t_2}{t_1}\left(dy^3+y^2\right)-N_{(a,b,c)}\frac{t_2^2}{t_1^3}y^3\right|$$
$$=\left|\left(\left(\frac{dt_1^2}{t_2}y\right)^2+D_{(a,b,c)}\right)\frac{dt_2}{t_1}y\right| = \begin{cases}\left|\frac{dt_2}{t_1}\right|, & \left|\frac{dt_1^2}{t_2}\right|\le 1\\ \left|\frac{d^3t_1^3}{t_2}\right|, & \left|\frac{dt_1^2}{t_2}\right|>1\end{cases}$$

and hence

$$\sum_{\substack{r\in\mathcal{O}/(\varpi)\\ r\not\equiv 0}}\Psi_E\left(gu\left(\frac{y^3r}{\varpi},\frac{y^2r}{\varpi},\frac{yr}{\varpi},\frac{r}{\varpi}\right)^{-1}g^{-1}\right)=\begin{cases}-1, & \left|\frac{dt_1^2}{t_2}\right|\le 1\&\left|\frac{dt_2}{t_1}\right|=1\\ & \left|\frac{dt_1^2}{t_2}\right|> 1\&\left|\frac{d^3t_1^3}{t_2}\right|=1\\ q-1, & \left|\frac{dt_1^2}{t_2}\right|\le 1\&\left|\frac{dt_2}{t_1}\right|<1\\ & \left|\frac{dt_1^2}{t_2}\right|> 1\&\left|\frac{d^3t_1^3}{t_2}\right|<1\end{cases}.$$

- <u>E is a field:</u> For $y \not\equiv 0$ it holds that

$$\left|\frac{t_1^3}{t_2}\left(d^3y^3+3d^2y^2+3dy+1\right)+D_{(a,b,c)}\frac{t_2}{t_1}\left(dy^3+y^2\right)-N_{(a,b,c)}\frac{t_2^2}{t_1^3}y^3\right|$$
$$=\left|(dy+1)^3\frac{t_1^3}{t_2}+D_{(a,b,c)}(dy+1)t^2\frac{t_2}{t_1}-N_{(a,b,c)}\frac{t_2^2}{t_1^3}y^3\right|$$
$$=\left|\left(\left(\frac{dy+1}{y}\frac{t_1^2}{t_2}\right)^3+D_{(a,b,c)}\frac{dy+1}{y}\frac{t_1^2}{t_2}-N_{(a,b,c)}\right)\frac{t_2^2}{t_1^3}\right| = \begin{cases}\left|\frac{t_2^2}{t_1^3}\right|, & \left|\frac{dt_1^2}{t_2}\right|\le 1\\ \left|\frac{d^3t_1^3}{t_2}\right|, & \left|\frac{dt_1^2}{t_2}\right|>1\end{cases}$$

and hence

$$\sum_{\substack{r\in\mathcal{O}/(\varpi)\\ r\not\equiv 0}}\Psi_E\left(gu\left(\frac{y^3r}{\varpi},\frac{y^2r}{\varpi},\frac{yr}{\varpi},\frac{r}{\varpi}\right)^{-1}g^{-1}\right)=\begin{cases}-1, & \left|\frac{dt_1^2}{t_2}\right|\le 1\&\left|\frac{t_2^2}{t_1^3}\right|=1\\ & \left|\frac{dt_1^2}{t_2}\right|> 1\&\left|\frac{d^3t_1^3}{t_2}\right|=1\\ q-1, & \left|\frac{dt_1^2}{t_2}\right|\le 1\&\left|\frac{t_2^2}{t_1^3}\right|<1\\ & \left|\frac{dt_1^2}{t_2}\right|> 1\&\left|\frac{d^3t_1^3}{t_2}\right|<1\end{cases}.$$

Combining all of the above yields the assertion for $d \notin \mathcal{O}$.

□

The explicit computation of $F^* (\Psi_E, \mathbb{1}, \cdot, s) * P_s (\cdot) (g)$ boils down to inserting Corollary 4.3.4 and Equation (4.18) into Equation (4.17), which is performed in Appendix A.

Chapter 5

Analytic Properties of $d_S(\chi, s, \Psi_E, \varphi, f)$

In this chapter we prove

Theorem 2.0.8 . *For any $s_0 \in \mathbb{C}$ there exist data φ_S and f_S such that $d_S(\chi, s, \Psi_E, \varphi_S, f_S)$ is entire and non-vanishing in a neighborhood of s_0.*

Proof. Let $\mu = \mu_E$. Recall from Theorem 2.0.3 that

$$d_S(\chi, s, \Psi_E, \varphi, f) = \int\limits_{U_S \backslash G_S} L_{\Psi_E}(\varphi)(g) F^*(\Psi_E, \chi, g, s)\, dg = \int\limits_{U_S^{\mu} \backslash G_S} L_{\Psi_E}(\varphi)(g) f_s^*(\mu g)\, dg .$$

For $g \in G_S^{\mu} = \mathrm{Stab}_{G(F_S)}(P_E(F_S)\mu)$ let

$$\chi_s^{\mu}(g) = \chi_s\left(\mu g \mu^{-1}\right) .$$

Since $P_E(F_S)\mu G(F_S)$ is open in $H_E(F_S)$ there is an inclusion

$$i : \mathrm{ind}_{G_S^{\mu}}^{G_S} \chi_s^{\mu} \hookrightarrow I_{P_E}(\chi, s)$$

given by

$$i(f_s)(h) = \begin{cases} \chi_s(p) f_s(g), & h = p\mu g \in P_E(F_S)\mu G_S,\ p \in P_E(F_S),\ g \in G_S \\ 0, & \text{otherwise} \end{cases} .$$

Let U_S^- denote the unipotent radical of the parabolic subgroup P^- of G_S opposite to P_S. We recall that $P_S \cdot U_S^-$ is open and dense in G_S homeomorphic to $P_S \times U_S^-$. On the other hand, recall that $G_S^\mu = U_S^\mu \cdot T_S^\mu$ where

$$U_S^\mu = \ker \Psi_{E,S},\ T_S^\mu = \left\{ h_{3\alpha+2\beta}(t) : t \in F_S^\times \right\}.$$

We note that

$$P_S = U_S^\mu \cdot T_S^\mu \cdot X_{3\alpha+\beta} \cdot \mathcal{Q}_S,$$

where

$$X_{3\alpha+\beta} = \{x_{3\alpha+\beta}(r) : r \in F_S\}, \quad \mathcal{Q}_S = \iota_\alpha(SL_2)(F_S).$$

For $\phi_S \in \mathcal{S}(F_S)$, $\Phi_S \in C_c^\infty\left(\mathcal{Q}_S \cdot U_S^-\right)$ we let

$$\widetilde{f}_s(g) = \begin{cases} \chi_s^\mu(g^\mu)\,\phi_S(r)\,\Phi_S(g_c), & g = g^\mu x_{3\alpha+\beta}(r)\,g_c \in P_S U_S^-, \\ & g^\mu \in G_S^\mu,\ r \in F_S,\ g_c \in \mathcal{Q}_S U_S^- \\ 0, & \text{otherwise} \end{cases}.$$

This is well defined since $P_S U_S^-$ is open-dense in G_S, ϕ_S and Φ_S are compactly supported.

Hence

$$\begin{aligned} \frac{d_S\left(\chi, s, \Psi_E, \varphi, i\left(\widetilde{f}_s\right)\right)}{j_E(\chi, s)} &= \int\limits_{U_S^\mu \backslash G_S} L_{\Psi_E}(\varphi)(g)\, i\left(\widetilde{f}_s\right)(\mu g)\, dg \\ &= \int\limits_{T_S^\mu} \int\limits_{F_S} \int\limits_{\mathcal{Q}_S U_S^-} L_{\Psi_E}(\varphi)\left(t_\mu x_{3\alpha+\beta}(r)\, g_c\right) \chi_s^\mu(t_\mu)\, \phi_S(r)\, \Phi_S(g_c)\ dg_c\ dr\ dt_\mu \\ &= \int\limits_{T_S^\mu} L_{\Psi_E}\left(\phi_S * (\Phi_S \star \varphi)\right)(t_\mu)\, \chi_s^\mu(t_\mu)\, dt_\mu\ , \end{aligned}$$

where

$$\Phi_S \star \varphi(g) = \int\limits_{\mathcal{Q}_S U_S^-} \varphi(g g_c)\, \Phi_S(g_c)\ dg_c$$

and

$$\phi_S * \varphi(g) = \int\limits_{F_S} \varphi\left(g x_{3\alpha+\beta}(r)\right) \phi_S(r)\ dr.$$

Lemma 5.0.1. $L_{\Psi_E}(\varphi)(\phi_S * \varphi)(h_{3\alpha+2\beta}(t)) = \widehat{\phi}_S(t)\, L_{\Psi_E}(\varphi)(h_{3\alpha+2\beta}(t))$.

Proof. It holds that

$$\begin{aligned}
L_{\Psi_E}(\phi_S * \varphi)(h_{3\alpha+2\beta}(t)) &= \int\limits_{U(F)\backslash U(\mathbb{A})} \phi_S * \varphi(u\ h_{3\alpha+2\beta}(t))\, \overline{\Psi_E(u)}\, du \\
&= \int\limits_{U(F)\backslash U(\mathbb{A})} \int\limits_{F_S} \phi_S(r)\, \varphi(u\ h_{3\alpha+2\beta}(t)\ \ x_{3\alpha+\beta}(r))\, dr\ \overline{\Psi_E(u)}\, du \\
&= \int\limits_{U(F)\backslash U(\mathbb{A})} \int\limits_{F_S} \phi_S(r)\, \varphi(u\ x_{3\alpha+\beta}(rt)\ \ h_{3\alpha+2\beta}(t))\, dr\ \overline{\Psi_E(u)}\, du \\
&= \int\limits_{U(F)\backslash U(\mathbb{A})} \left(\int\limits_{F_S} \phi_S(r)\, \psi(rt)\, dr \right) \varphi(u\ h_{3\alpha+2\beta}(t))\, \overline{\Psi_E(u)}\, du \\
&= \widehat{\phi}_S(t)\, L_{\Psi_E}(\varphi)(h_{3\alpha+2\beta}(t))\, .
\end{aligned}$$

□

Writing $t_\mu = h_{3\alpha+2\beta}(t)$ yields

$$\frac{d_S\left(\chi, s, \Psi_E, \varphi, i\left(\widetilde{f}_s\right)\right)}{j_E(\chi, s)} = \int\limits_{F_S^\times} \widehat{\phi_S}(t)\, \chi(t)\, |t|^{s+\frac{1}{2}}\, L_{\Psi_E}(\Phi_S \star \varphi)(h_{3\alpha+2\beta}(t))\ \ dt^\times .$$

Note that $\mathcal{Q}_S U_S^-$ is a subgroup of G_S. By the assumption the π supports the Ψ_E-Fourier coefficient there exist $\varphi = i\left(\underset{\nu\in\mathcal{P}}{\otimes} \varphi_\nu\right) \in \pi$ such that φ_ν is unramified for all $\nu \notin S$ and $L_{\Psi_E}(\varphi) \not\equiv 0$. By the Dixmier-Malliavin Theorem [10], applied to π as a representation of $\mathcal{Q}_S U_S^-$, there exist Φ_S and $\varphi \in \pi^\infty$ such that $L_{\Psi_E}(\Phi_S \star \varphi) \not\equiv 0$.

We note that by the Paley-Wiener Theorem $\widehat{\phi_S} \in \mathcal{S}(F_S)$. Since the image of $F_S^\times$ inside F_S is locally closed, we may choose ϕ_S such that $0 \notin supp\left(\widehat{\phi_S}\right)$, ensuring the entireness of $d_S\left(\chi, s, \Psi_E, \varphi, i\left(\widetilde{f}_s\right)\right)$. Fixing $t_0 \in F_S^\times$ such that $L_{\Psi_E}(\Phi_S \star \varphi)(h_{3\alpha+2\beta}(t)) \neq 0$ we choose ϕ_S such that $\widehat{\phi_S}$ is of compact support containing t_0 and such that $d_S\left(\chi, s_0, \Psi_E, \varphi, i\left(\widetilde{f}_s\right)\right) \neq 0$. We note that $i\left(\widetilde{f}_s\right)$ may not be K-finite, however by the the density of the K-finite vectors in the smooth vectors one may choose f_s that satisfy the required properties.

□

Corollary 5.0.2. *For any choice of $\varphi \in \pi$ and K-finite section f_s the function $d_S(\chi, s, \Psi_E, \varphi, f)$ admits a meromorphic continuation in s.*

Proof. This follows immediately from Theorem 2.0.2 and the fact that $\mathcal{Z}_E(\chi, s, \varphi, f)$ is meromorphic. □

Remark 5.0.3. Note that for general choices of φ and f_s the function $d_S(\chi, s, \Psi_E, \varphi, f)$ need not be holomorphic and could, in fact, have poles.

Chapter 6

The Poles of the Eisenstein Series

We say that a complex function h admits a pole of order n at s_0 if

$$\lim_{s\to s_0} (s-s_0)^n\, h(s) \in \mathbb{C}^\times.$$

We say that $\mathcal{E}_E(\chi,\cdot,s,\cdot)$ admits a pole of order n at s_0 if

$$\sup\left\{\mathrm{ord}_{s=s_0}\, \mathcal{E}_E(\chi,f_s,s,g) : f_s\in I_{P_E}(\chi,s),\ g\in H_E(\mathbb{A})\right\} = n. \tag{6.1}$$

We show that for $\mathfrak{Re}(s)>0$ the order is bounded.

Remark 6.0.1. Note that this is not necessarily so, since even in the case of the Eisenstein series of SL_2, the pole at $s=-\frac{1}{2}$ is of unbounded order, in the sense that

$$\forall n\in\mathbb{Z}\quad \exists f_s\in \mathrm{Ind}_{\mathcal{B}}^{SL_2}\,\delta_{\mathcal{B}}^{s+\frac{1}{2}},\ g\in SL_2(\mathbb{A}):\quad \mathrm{ord}_{s=-\frac{1}{2}}\,\mathcal{E}_{SL_2}(f_s,s,g)\geq n.$$

In this chapter, we analyze the possible poles of $\mathcal{E}_E(\chi,f_s,s,g)$ in the half-plane $\mathfrak{Re}(s)>0$ and their orders. This is done in Theorem 6.2.1.

6.1 Background Theory on Eisenstein Series and Intertwining Operators

We start by recalling some general information regarding the theory of Eisenstein series. Most of the results quoted in this section can be found in [34].

6.1.1 Intertwining Operators and the Constant Term

We start by noting that

$$I_{P_E}(\chi, s) \hookrightarrow I_{B_E}(\chi_s) = \mathrm{Ind}_{B_E}^{H_E} \delta_{B_E}^{\frac{1}{2}} \chi_s, \tag{6.2}$$

where

$$\chi_s = \delta_{B_E}^{-\frac{1}{2}} \otimes (\chi \circ \det_{M_E}) \otimes |\det_{M_E}|^{s+\frac{5}{2}}.$$

Note that the induction here is unnormalized. We also note that in this chapter we use the notation χ_s for a different character than that in Chapter 4 or Chapter 5.

For any $w \in W$ we define the intertwining operator

$$M(w, \chi_s) : I_{B_E}(\chi_s) \to I_{B_E}\left(w^{-1} \cdot \chi_s\right)$$

by

$$M(w, \chi_s) f_s(g) = \int\limits_{N_E(\mathbb{A}) \cap w^{-1} N_E(\mathbb{A}) w \backslash N_E(\mathbb{A})} f_s(wng)\, dn. \tag{6.3}$$

When there is no source of confusion we denote $M(w, \chi_s)$ by M_w. We also denote $w_{i_1,\dots,i_k}$ for $w_{\alpha_{i_1}} \cdots w_{\alpha_{i_k}}$, where $w_{\alpha_i} \in W_{H_E}$ denotes the simple reflection associated with the simple root α_i.

Remark 6.1.1. Note that the definition here is slightly different from the definition given in [34]. As a consequence, if $w = w'w''$ is a reduced word then

$$M_{w,\chi_s} = M_{w'',w'^{-1}\cdot\chi_s} \circ M_{w',\chi_s}.$$

The constant term of $\mathcal{E}_E(\chi, f_s, s, g)$ along T_E is defined to be

$$\mathcal{E}_E(\chi, f_s, s, g)_{T_E} = \int\limits_{N_E(F) \backslash N_E(\mathbb{A})} \mathcal{E}_E(\chi, f_s, s, ug)\, du \quad \forall g \in T_E. \tag{6.4}$$

By a standard computation, as in [22], we obtain

$$\mathcal{E}_E(\chi, f_s, s, g)_{T_E} = \sum_{w \in W(P_E, H_E)} M_w(s) f_s|_{T_E}(g) \quad \forall g \in T_E, \tag{6.5}$$

where $W(P_E, H_E) = \{w \in W_{H_E} : w(\alpha_2) > 0,\ w^{-1}(\alpha_i) > 0 \text{ for } i = 1, 3, 4\}$ is a set of distinguished representatives for $P_E \backslash H_E / B_E$ by the shortest elements in each coset.

Theorem 6.1.2. *The degenerate Eisenstein series $\mathcal{E}_E(\chi, f_s, s, g)$ admits a pole of order n at (χ, s_0) if and only if its constant term $\mathcal{E}_E(\chi, f_s, s, g)_{T_E}$ admits a pole of order n at (χ, s_0).*

Indeed, in Section 6.2 we study the poles of $\mathcal{E}_E(\chi, f_s, s, g)$ via the poles of $\mathcal{E}_E(\chi, f_s, s, g)_{T_E}$, using Equation (6.5).

6.1.2 Rank-one Intertwining Operators and Local Factors

In many instances, the study of Eisenstein series and intertwining operators relies on reduction to the rank-one case. In this section, we recall some useful facts about the rank-one case and the reduction to it.

We fix a number field L and let D_L be the discriminant of $L/\mathbb{Q}$ and let $\zeta_L(s)$ be the completed ζ-function of L. Following [26], we define

$$\xi_L(s) = |D_L|^{\frac{s}{2}} \zeta_L(s).$$

The normalized function ξ_L then satisfies the functional equation

$$\xi_L(s) = \xi_L(1-s). \tag{6.6}$$

Let $\mathcal{B} = \mathcal{T} \cdot \mathcal{N}$ be the Borel subgroup of SL_2 with torus $\mathcal{T}$ and unipotent radical $\mathcal{N}$. Also let $w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be the generator of the Weyl group of SL_2. We recall some facts about the intertwining operator M_{w_0} defined on representations of SL_2. Fix a Hecke character $\sigma = \underset{\nu\in\mathcal{P}}{\otimes} \sigma_\nu$ of $T(\mathbb{A})$, it can be considered as a representation of $\mathcal{B}$. For a section $f_s \in \operatorname{Ind}_{\mathcal{B}}^{SL_2} \sigma\delta_{\mathcal{B}}^{s+\frac{1}{2}}$ we let

$$M(w_0, \sigma, s) f_s(g) = \int\limits_{\mathcal{N}(\mathbb{A})\cap w_0^{-1}\mathcal{N}(\mathbb{A})w_0\backslash\mathcal{N}(\mathbb{A})} f_s(w_0 n g)\, dn = \int\limits_{\mathcal{N}(\mathbb{A})} f_s(w_0 n g)\, dn. \tag{6.7}$$

This integral converges for $\mathfrak{Re}(s) \gg 0$ and admits meromorphic continuation to the whole complex plane. The intertwining operator M_w is factorizable in the sense that if $f_s = \otimes f_{s,\nu}$ then $M(w_0,\sigma,s) f_s = \underset{\nu\in\mathcal{P}}{\otimes} M(w_0,\sigma_\nu,s) f_{s,\nu}$, where

$$M(w_0,\sigma_\nu,s) f_{s,\nu}(g) = \int\limits_{\mathcal{N}(F_\nu)} f_{s,\nu}(w_0 n g)\, dn. \tag{6.8}$$

For a spherical section f_ν^0 of $\mathrm{Ind}_{\mathcal{B}(F_\nu)}^{SL_2(F_\nu)} \sigma_\nu \delta_{\mathcal{B}}^{s+\frac{1}{2}}$ it holds that

$$M(w_0,\sigma_\nu,s) f_{s,\nu}^0 = \frac{\mathcal{L}_{F_\nu}(2s,\sigma_\nu)}{\mathcal{L}_{F_\nu}(2s+1,\sigma_\nu)} f_{-s,\nu}^0, \tag{6.9}$$

where:

- For $\nu \not| \infty$
$$\mathcal{L}_{F_\nu}(s,\sigma_\nu) = \frac{1}{1-\sigma_\nu(\varpi_\nu)\, q_\nu^{-s}},$$
for a uniformizer ϖ_ν of L_ν and q_ν the cardinality of the residue field of F_ν. This function is a non-vanishing meromorphic function on $\mathbb{C}$ with a simple poles at $s = \frac{\log(\sigma_\nu(\varpi_\nu))+2\pi i n}{\log(q_\nu)}$ for all $n\in\mathbb{Z}$.

- The only finite order characters σ_ν of $\mathbb{R}^\times$ are either the trivial one or the sign character. Let
$$\epsilon_\nu = \begin{cases} 0, & \sigma_\nu = \mathbb{1} \\ 1, & \sigma_\nu = \mathrm{sgn} \end{cases}$$
and
$$\mathcal{L}_{\mathbb{R}}(s,\sigma_\nu) = \pi^{-\frac{s+\epsilon_\nu}{2}} \Gamma\left(\frac{s+\epsilon_\nu}{2}\right).$$

- The only finite order character σ_ν of $\mathbb{C}^\times$ is the trivial one. For $n\in\mathbb{Z}$ let
$$\sigma_{n,\nu}(z) = \left(\frac{z}{|z|}\right)^n.$$
Note that any continuous complex character of $\mathbb{C}^\times$ is of the form $\sigma_n(z)\,|z|^s$ for some $n\in\mathbb{Z}$ and $s\in\mathbb{C}$. Let
$$\mathcal{L}_{\mathbb{C}}(s,\sigma_{n,\nu}) = (2\pi)^{-\left(s+\frac{|n|}{2}\right)} \Gamma\left(s+\frac{|n|}{2}\right).$$

Recall that $\Gamma(z)$ is a non-vanishing meromorphic function on $\mathbb{C}$ whose only poles are simple appearing at the points $z=-n$ for $n\geq 0$.

Studying the analytic behavior of $M(w_0,\sigma_\nu,s)$, we have the following lemma ([46] for $\nu\not|\infty$ and [42] for $\nu|\infty$):

Lemma 6.1.3. *For any $\sigma_\nu : F_\nu^\times \to \mathbb{C}^\times$ the operator $\frac{1}{\mathcal{L}_L(2s,\sigma_\nu)} M(w_0,\sigma_\nu,s)$ is entire.*

For $\nu\not|\infty$ it holds that (from the above and [44, Section 11]):

- The operator $M_\nu(w_0,\sigma_\nu,s)$ is entire for σ_ν ramified.
- If σ_ν is unramified then $M_\nu(w_0,\sigma_\nu,s)$ is meromorphic with a simple poles at $\frac{\log(\sigma_\nu(\varpi_\nu))+2\pi i n}{\log(q_\nu)}$ for all $n\in\mathbb{Z}$.
- Furthermore, when $\sigma_\nu=\mathbb{1}$ then $M_\nu(w_0,\sigma_\nu,s)$ is not injective at $s=\frac{1}{2}$ and $s=-\frac{1}{2}$.
- Furthermore, when $\sigma_\nu=\mathbb{1}$ then $M_\nu(w_0,\sigma_\nu,s)$ is not injective at $s=\frac{1}{2}$ and $s=-\frac{1}{2}$.
- In particular, for $\sigma_\nu=1$ we have

$$0\longrightarrow 1\longrightarrow \mathrm{Ind}_{\mathcal{B}}^{SL_2} 1 \xrightarrow{M_{w_0}} St\longrightarrow 0$$
$$0\longrightarrow St\longrightarrow \mathrm{Ind}_{\mathcal{B}}^{SL_2} \delta_{\mathcal{B}}^1 \xrightarrow{M_{w_0}} 1\longrightarrow 0.$$

- Whenever $\sigma_\nu\neq\mathbb{1}$, the induced representation $\mathrm{Ind}_{\mathcal{B}}^{SL_2}\sigma\delta_{\mathcal{B}}^{s+\frac{1}{2}}$ is reducible if and only if $\sigma_\nu^2=\mathbb{1}$ and $s=0$. In this case, $\mathrm{Ind}_{\mathcal{B}}^{SL_2}\sigma\delta_{\mathcal{B}}^{s+\frac{1}{2}}=\pi_\nu^{(1)}\oplus\pi_\nu^{(-1)}$ where $\pi_\nu^{(1)}$ and $\pi_\nu^{(-1)}$ are irreducible and if σ_ν is unramified then $\pi_\nu^{(1)}$ is also unramified. Furthermore, $M_\nu(w_0,\sigma_\nu,0)$ is bijective and acts as multiplication by a scalar on $\pi_\nu^{(1)}$ and $\pi_\nu^{(-1)}$.

We now discuss the case $\nu|\infty$ (from the above and [30, Chapters II and VII]).
If $F_\nu=\mathbb{R}$ then $\Pi_{\epsilon_\nu,s}=\mathrm{Ind}_{\mathcal{B}}^{SL_2}\sigma_\nu\delta_B^{s+\frac{1}{2}}$ is reducible if and only if $2s=n\in\mathbb{Z}$ and

$$\epsilon_\nu\equiv n+1 \mod 2$$

In which case, the decomposition series for $\Pi_{\epsilon_\nu,s}$ is as follows:

- For $s=0$ it holds that
$$\Pi_{\epsilon_\nu,s}=\mathcal{D}_1^+\oplus\mathcal{D}_1^-,$$
where $\mathcal{D}_1^+$ and $\mathcal{D}_1^-$ are irreducible representations known as the holomorphic and non-holomorphic limits of discrete series (respectively).

- For $2s=n\in\mathbb{N}$ it holds that
$$\mathcal{D}_{n-1}^+\oplus\mathcal{D}_{n-1}^-\hookrightarrow\Pi_{\epsilon_\nu,s}\twoheadrightarrow\Phi_{n-1},$$
where Φ_{n-1} is the unique irreducible representation of $SL_2(\mathbb{R})$ of dimension $n-1$ and $\mathcal{D}_{n-1}^+$, $\mathcal{D}_{n-1}^-$ are the irreducible representations known as the holomorphic and non-holomorphic discrete series of highest weight $n-1$.

- For $-2s=n\in\mathbb{N}$ it holds that
$$\Phi_{n-1}\hookrightarrow\Pi_{\epsilon_\nu,s}\twoheadrightarrow\mathcal{D}_{n-1}^+\oplus\mathcal{D}_{n-1}^-.$$

If $F_\nu=\mathbb{C}$ then $\Pi_{n,s}=\mathrm{Ind}_{\mathcal{B}}^{SL_2}\sigma_{n,\nu}\left|\cdot\right|^s$ is reducible if and only if $n=l-k$ and $4s=2+k+l$ or $n=k-l$ and $4s=-(2+k+l)$ for $k,l\in\mathbb{N}\cup\{0\}$, in which case

- If $n=l-k$ and $4s=2+k+l$ then
$$\Phi_{k,l}\hookrightarrow\Pi_{n,s}\twoheadrightarrow\mathcal{E}_{n-1}^+\oplus\mathcal{E}_{n-1}^-,$$
where $\Phi_{k,l}$ is the finite-dimensional representation realized as polynomials in the complex variables $(z_1,z_2,\overline{z_1},\overline{z_2})$, homogeneous of degree k in (z_1,z_2) and homogeneous of degree l in $(\overline{z_1},\overline{z_2})$. $\mathcal{E}_{n-1}^+$ and $\mathcal{E}_{n-1}^-$ are analogous to $\mathcal{D}_{n-1}^+$ and $\mathcal{D}_{n-1}^-$.

We finish the discussion of the case of SL_2 by recalling a result regarding the global intertwining operator. As $M(w_0,s,\sigma)\circ M(w_0,-s,\sigma)$ is an endomorphism of irreducible representations for all $s\neq\pm\frac{1}{2}$, it equals a constant.

Lemma 6.1.4 ([32], Lemma 6.3)**.** *It holds that*
$$M(w_0,s,\sigma)\circ M(w_0,-s,\overline{\sigma})=\mathrm{Id}\quad\forall s\in\mathbb{C}.$$
For $s=\pm\frac{1}{2}$ this should be understood as
$$\lim_{s\to\frac{1}{2}}M(w_0,s,\sigma)\circ M(w_0,-s,\overline{\sigma})=\mathrm{Id}\,.$$

6.1.3 Intertwining Operators for Induced Representations of H_E

At this point, it will be beneficial to consider a more general point of view. Let $\mathfrak{a}_{\mathbb{C}}^* = X^*(T_E) \otimes \mathbb{C}$. We equip $\mathfrak{a}_{\mathbb{C}}^{\times}$ with the following system of coordinates:

- If $E = F \times F \times F$ we have $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}^4$ and we write $\lambda = (s_1, s_2, s_3, s_4) \in \mathfrak{a}_{\mathbb{C}}^*$ for

$$\lambda \left(h_{\alpha_1}(t_1) h_{\alpha_2}(t_2) h_{\alpha_3}(t_3) h_{\alpha_4}(t_4)\right) = |t_1|_F^{s_1} |t_2|_F^{s_2} |t_3|_F^{s_3} |t_4|_F^{s_4} \quad \forall t_1, t_2, t_3, t_4 \in F^{\times}.$$

- If $E = F \times K$ we have $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}^3$ and we write $\lambda = (s_1, s_2, s_3) \in \mathfrak{a}_{\mathbb{C}}^*$ for

$$\lambda \left(h_{\alpha_1}(t_1) h_{\alpha_2}(t_2) h_{\alpha_3}(t_3) h_{\alpha_4}(t_3^{\sigma})\right) = |t_1|_F^{s_1} |t_2|_F^{s_2} |t_3|_K^{s_3} \quad \forall t_1, t_2 \in F^{\times},\ \forall t_3 \in K^{\times}.$$

- If E is a field we have $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}^2$ and we write $\lambda = (s_1, s_2) \in \mathfrak{a}_{\mathbb{C}}^*$ for

$$\lambda \left(h_{\alpha_1}(t_1) h_{\alpha_2}(t_2) h_{\alpha_3}(t_1^{\sigma}) h_{\alpha_4}\left(t_1^{\sigma^2}\right)\right) = |t_1|_E^{s_1} |t_2|_F^{s_2} \quad \forall t_2 \in F^{\times},\ \forall t_1 \in E^{\times}.$$

For any finite order character $\chi = \underset{\nu \in \mathcal{P}}{\otimes} \chi_\nu$ of $T_E(\mathbb{A})$ and any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we let

$$\begin{aligned} I_{B_E}(\chi, \lambda) &= \mathrm{Ind}_{B_E(\mathbb{A})}^{H_E(\mathbb{A})} (\chi \circ det_{M_E}) \cdot (\lambda + \rho_{B_E}) = \underset{\nu \in \mathcal{P}}{\otimes} I_{B_E}(\chi_\nu, \lambda) \\ I_{B_E}(\chi_\nu, \lambda) &= \mathrm{Ind}_{B_E(F_\nu)}^{H_E(F_\nu)} (\chi_\nu \circ det_{M_E}) \cdot (\lambda + \rho_{B_E}), \end{aligned}$$

where ρ_{B_E} is half the sum of positive roots in H_E with respect to B_E. This is not the most general principal series representation but it will suffice for our needs. We note that

$$I_{P_E}(\chi, s) \hookrightarrow I_{B_E}(\chi_s) = I_{B_E}(\chi, \lambda_s),$$

where

$$\lambda_s = \begin{cases} \left(-1, s + \frac{3}{2}, -1, -1\right), & E = F \times F \times F \\ \left(-1, s + \frac{3}{2}, -1\right), & E = F \times K \\ \left(-1, s + \frac{3}{2}\right) & E/F \text{ is a cubic field extension} \end{cases}. \tag{6.10}$$

For $w \in W$ and $f_\lambda \in I_B(\chi, \lambda)$ let

$$M(w, \chi, \lambda) f_\lambda(g) = \int\limits_{N_E(\mathbb{A}) \cap w^{-1} N_E(\mathbb{A}) w \backslash N_E(\mathbb{A})} f_\lambda(wng)\, dn. \tag{6.11}$$

This integral converges absolutely to an analytic function on the positive Weyl chamber

$$C^+ = \{\lambda \in \mathfrak{a}_\mathbb{C}^* : \mathfrak{Re}\langle \lambda, \alpha^\vee \rangle > 0\ \forall \alpha > 0\}$$

and admits a meromorphic continuation to $\mathfrak{a}_\mathbb{C}^*$.

Remark 6.1.5. Due to the choice of representatives in $W(P_E, H_E)$, the intertwining operators $M(w, \chi_s)$ defined in Equation (6.3) are generically (at points of holomorphicity) restrictions of $M(w, \chi, \lambda)$ to the line λ_s as above.

Note that by abuse of notation, for a Hecke character χ we identify χ and $\chi \circ det_{M_E}$.

We first recall that $M(w, \chi, \lambda)$ and $M(w, \chi_s)$ can be decomposed as follows

$$\begin{aligned} M(w, \chi_\nu, \lambda) &= \underset{\nu \in \mathcal{P}}{\otimes} M_\nu(w, \chi_\nu, \lambda) \\ M(w, \chi_s) &= \underset{\nu \in \mathcal{P}}{\otimes} M_\nu(w, \chi_{\nu,s}), \end{aligned} \tag{6.12}$$

where for any $\nu \in \mathcal{P}$, $\lambda \in C^+$ and $\mathfrak{Re}(s) \gg 0$, the local intertwining operators $M(w, \chi_\nu, \lambda)$ and $M_\nu(w, \chi_s)$ are defined via

$$\begin{aligned} M(w, \chi_\nu, \lambda) f_{\lambda,\nu}(g) &= \int\limits_{N_E(F_\nu) \cap w^{-1} N_E(F_\nu) w \backslash N_E(F_\nu)} f_{\lambda,\nu}(wng)\, dn \\ M(w, \chi_{s,\nu}) f_{s,\nu}(g) &= \int\limits_{N_E(F_\nu) \cap w^{-1} N_E(F_\nu) w \backslash N_E(F_\nu)} f_{s,\nu}(wng)\, dn. \end{aligned} \tag{6.13}$$

We recall the connection between the rank-one case and the intertwining operators M_{w_α}, where w_α is the simple reflection with respect to a simple root α. For any simple root α, we have an embedding $\iota_\alpha : SL_2 \to H_E$, defined over F_α, so that

$$\iota_\alpha\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = x_\alpha(x), \quad \iota_\alpha\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) = x_{-\alpha}(x), \quad \iota_\alpha\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = w_\alpha.$$

Lemma 6.1.6. *The following diagram is commutative*

$$\begin{array}{ccc} I_{B_E}(\chi_\nu,\lambda) & \xrightarrow{M_\nu(w_\alpha,\chi\nu,\lambda)} & I_{B_E}(w_\alpha\cdot\chi_\nu, w_\alpha\cdot\lambda) \\ \iota_\alpha^* \downarrow & & \downarrow \iota_\alpha^* \\ \mathrm{Ind}_{\mathcal{B}(F_\nu)}^{SL_2(F_\nu)}(\chi_\nu|_{T_E}\otimes\lambda) & \xrightarrow{M_{w_0}} & \mathrm{Ind}_{\mathcal{B}(F_\nu)}^{SL_2(F_\nu)}(w_\alpha\cdot\chi_\nu|_{T_E}\otimes\lambda) \end{array},$$

where the vertical maps should be understood as the pull-back map. By restriction to $I_{P_E}(\chi,s)$, this is also true for $M(w_\alpha,\chi_s)$.

Proof. We note that

$$\iota_\alpha\left(N_E(F_\nu)\cap w_\alpha^{-1}N_E(F_\nu)\,w_\alpha\backslash N_E(F_\nu)\right)=\mathcal{N}(F_\nu)\cap w_0^{-1}\mathcal{N}(F_\nu)\,w_0\backslash N_E(F_\nu)\cong\mathcal{N}(F_\nu)\,.$$

Consequently, for $f_{s,\nu}\in I_{B_E}(\chi_{\nu,s})$ and $g\in SL_2(F_\nu)$ it holds that

$$\begin{aligned} M_{w_0}\iota_\alpha^*(f_{s,\nu})(g) &= \int\limits_{\mathcal{N}(F_\nu)} \iota_\alpha^*(f_{s,\nu})(w_0ng)\,dn \\ &= \int\limits_{\mathcal{N}(F_\nu)} (f_{s,\nu})(\iota_\alpha(w_0ng))\,dn \\ &= \int\limits_{N_E(F_\nu)\cap w^{-1}N_E(F_\nu)w\backslash N_E(F_\nu)} (f_{s,\nu})(w_\alpha n'\iota_\alpha(g))\,dn' \\ &= (M_\nu(w_\alpha,\chi_{\nu,s})f)(\iota_\alpha(g)) = \iota_\alpha^*(M_\nu(w_\alpha,\chi_{\nu,s})f)(g)\,. \end{aligned}$$

□

Following is a corollary of the previous lemma and Equation (6.9).

Corollary 6.1.7 (The Gindikin-Karpelevich formula)**.** *Let $\nu\in\mathcal{P}$ be a place such that χ_ν is unramified. Also, let $w\in W$.*

- *Let $f_\nu^0\in I_{B_E}(\chi_\nu,\lambda)$ be an unramified vector. It then holds that*

$$M_\nu(w,s,\chi_\nu)\,f_\nu^0=\prod_{\alpha>0,\ w^{-1}\alpha<0}\frac{\mathcal{L}_{F_{\alpha,\nu}}\left(\langle\lambda,\alpha^\vee\rangle,\chi_\nu\circ\det_{M_E}\circ\alpha^\vee\right)}{\mathcal{L}_{F_{\alpha,\nu}}\left(\langle\lambda,\alpha^\vee\rangle+1,\chi_\nu\circ\det_{M_E}\circ\alpha^\vee\right)}f_\nu^0. \tag{6.14}$$

- *Let* $f_\nu^0 \in I_{B_E}(\chi_{s,\nu})$ *be an unramified vector. It then holds that*

$$M_\nu(w,s,\chi_\nu) f_\nu^0 = \prod_{\alpha>0,\ w^{-1}\alpha<0} \frac{\mathcal{L}_{F_{\alpha,\nu}}(\chi_{s,\nu}\circ\alpha^\vee)}{\mathcal{L}_{F_{\alpha,\nu}}\left(q_{\alpha,\nu}^{-1}\chi_{s,\nu}\circ\alpha^\vee\right)} f_\nu^0. \tag{6.15}$$

We denote the Gindikin-Karpelevich term by

$$\begin{aligned} J_\nu(w,\chi,\lambda) &= \prod_{\alpha>0,\ w^{-1}\alpha<0} \frac{\mathcal{L}_{F_{\alpha,\nu}}(\langle\lambda,\alpha^\vee\rangle,\chi_\nu\circ\det_{M_E}\circ\alpha^\vee)}{\mathcal{L}_{F_{\alpha,\nu}}(\langle\lambda,\alpha^\vee\rangle+1,\chi_\nu\circ\det_{M_E}\circ\alpha^\vee)} f_\nu^0 \\ J_\nu(w,\chi_s) &= \prod_{\alpha>0,\ w^{-1}\alpha<0} \frac{\mathcal{L}_{F_{\alpha,\nu}}(\chi_{s,\nu}\circ\alpha^\vee)}{\mathcal{L}_{F_{\alpha,\nu}}\left(q_{\alpha,\nu}^{-1}\chi_{s,\nu}\circ\alpha^\vee\right)}. \end{aligned} \tag{6.16}$$

We list the various Gindikin-Karpelevich terms and their poles in Table B.1, Table B.5 and Table B.9.

We recall a corollary of [29, Proposition 6.3] which also follows from [26, Lemma 1.5].

Corollary 6.1.8. *Let* w_α *be a simple reflection and let* σ *be a unitary character of* $T(\mathbb{A})$ *so that* $w_\alpha\cdot\sigma=\sigma$ *then*

$$M_{w_\alpha} : \mathrm{Ind}_{B_E}^{H_E}(\sigma) \to \mathrm{Ind}_{B_E}^{H_E}(\sigma)$$

acts as a scalar multiplication by -1.

We would like to understand the poles of $M(w,\chi_s)$ in order to study the poles of $\mathcal{E}_E(\chi,f_s,s,g)$.

Let $f_s \in I_{P_E}(\chi,s)$ be a factorizable section and write $f_s = \underset{\nu\in\mathcal{P}}{\otimes} f_{s,\nu}$ and let $S\subset\mathcal{P}$ denote a finite set of places such that $f_{s,\nu}$ is the unique normalized spherical vector for $\nu\notin S$.

Denote

$$J(w,\chi_s) = \prod_{\nu\in\mathcal{P}} J_\nu(w,\chi_s); \quad J^S(w,\chi_s) = \prod_{\nu\notin S} J_\nu(w,\chi_s). \tag{6.17}$$

By Corollary 6.1.7, it holds that

$$\begin{aligned} M(w,\chi_s) f_s &= \left(\underset{\nu\in S}{\otimes} M_\nu(w,\chi_s) f_{s,\nu}\right)\otimes\left(\underset{\nu\notin S}{\otimes} J_\nu(w,\chi_s) f_{s,\nu}\right) = \\ & J(w,\chi_s)\left(\underset{\nu\in S}{\otimes} J_\nu(w,\chi_s)^{-1} M_\nu(w,\chi_s) f_{s,\nu}\right)\otimes\left(\underset{\nu\notin S}{\otimes} f_{s,\nu}\right). \end{aligned} \tag{6.18}$$

Hence the analytic behavior of $M(w,\chi_s)f_s$ is governed by that of $J(w,\chi_s)$ and $J_\nu(w,\chi_s)^{-1}M_\nu(w,\chi_s)f_{s,\nu}$ for $\nu \in S$. Note that according to Lemma 6.1.3, $J_\nu(w,\chi_s)^{-1}M_\nu(w,\chi_s)f_{s,\nu}$ is holomorphic whenever $\mathfrak{Re}(\langle\chi_s,\alpha^\vee\rangle) > -1$ for all $\alpha > 0$ such that $w \cdot \alpha < 0$. In light of Tables B.3, B.7, and B.11 and the discussion in Section 6.1.2 the following holds:

Lemma 6.1.9. *For any $\mathfrak{Re}(s_0) > 0$ and $\nu \in \mathcal{P}$ it holds that $J_\nu(w,\chi_s)^{-1}M_\nu(w,\chi_s)f_{s,\nu}$ is analtyic at s_0. Moreover, there exists an $f_{s,\nu}$ such that $J_\nu(w,\chi_s)^{-1}M_\nu(w,\chi_s)f_{s,\nu}$ is non-zero at s_0.*

6.2 Poles of the Eisenstein Series

In this section we make use of Equation (6.5) to study the poles of $\mathcal{E}_E(\chi,f_s,s,g)$ through the poles of $\mathcal{E}_E(\chi,f_s,s,g)_{T_E}$. We start by considering the poles of the various intertwining operators, thus getting a bound on the order of the poles. In the following table we list the possible poles of the Eisenstein series and bounds on the orders at these points. In the case that $E = F \times K$ and K is a field we denote $\chi_E = \chi_K$. In the case $E = F \times F \times F$ it holds that $\chi_E = \mathbb{1}$. In the case that E is a non-Galois field extension, class field theory does not apply and hence χ_E is not defined.

	$s=\frac{1}{2}$			$s=\frac{3}{2}$		$s=\frac{5}{2}$
	$\chi=\mathbb{1}$	$\chi=\chi_E$	χ quad.	$\chi=\mathbb{1}$	$\chi=\chi_E$	$\chi=\mathbb{1}$
$E=F\times F\times F$	4		1	3		1
$E=F\times K$	3	2	1	2	1	1
E Galois field extension	2	1	1	1	1	1
E non-Galois field extension	2	-	1	1	-	1

Table 6.1: Trivial Bounds on the Order of Poles of $\mathcal{E}_E(\chi,f_s,s,g)$

Theorem 6.2.1. *The orders of the poles of $\mathcal{E}_E(\chi, \cdot, s, \cdot)$ for $\mathfrak{Re}(s) > 0$ are bounded by the following numbers:*

	$s=\frac{1}{2}$			$s=\frac{3}{2}$		$s=\frac{5}{2}$
	$\chi = \mathbb{1}$	$\chi = \chi_E$	χ quad.	$\chi = \mathbb{1}$	$\chi = \chi_E$	$\chi = \mathbb{1}$
$E = F \times F \times F$	1		1	2		1
$E = F \times K$	1	2	1	1	1	1
E Galois field extension	1	1	1	0	1	1
E non-Galois field extension	1	-	1	0	-	1

Table 6.2: Bounds on the Order of Poles of $\mathcal{E}_E(\chi, f_s, s, g)$

Moreover, when $\chi = \mathbb{1}$, a pole of the above-mentioned order is obtained for the spherical vector. In the cases where $s = \frac{3}{2}$ and $\chi = \chi_E$ these orders can also be realized and the leading coefficient in the Laurent series generates the minimal representation of $H_E(\mathbb{A})$.

Furthermore, the residual representation of $\mathcal{E}_E(\chi, \cdot, s, \cdot)$ at the following points is square-integrable:

1. *When E is a field, at*
   - $s = \frac{1}{2}$ *and* $\chi^2 = \mathbb{1}$ *or* $\chi = \chi_E$ *(if E is a Galois field extension of F).*
   - $s = \frac{3}{2}$ *and* $\chi = \mathbb{1}$.
   - $s = \frac{5}{2}$ *and* $\chi = \mathbb{1}$.
2. *When $E = F \times K$ and K is a field, at*
   - $s = \frac{1}{2}$ *and* $\chi^2 = \mathbb{1}$ *with* $\chi \neq \mathbb{1}, \chi_K$.
   - $s = \frac{3}{2}$ *and* $\chi = \chi_K$.
   - $s = \frac{5}{2}$ *and* $\chi = \mathbb{1}$.

3. *When* $E = F \times F \times F$, *at*

   - $s = \frac{1}{2}$ *and* $\chi^2 = \mathbb{1}$ *with* $\chi \neq \mathbb{1}$.
   - $s = \frac{3}{2}$ *and* $\chi = \mathbb{1}$.
   - $s = \frac{5}{2}$ *and* $\chi = \mathbb{1}$.

The proof in the cases where E is not a field, $s = \frac{1}{2}$ and $\chi = \mathbb{1}$ assumes that the global degenerate principal series $I_{P_E}\left(\mathbb{1}, \frac{1}{2}\right)$ is generated by the global spherical vector. The global representation $I_{P_E}(\chi, s)$ is generated by the spherical vector if and only if $I_{P_E}(\chi_\nu, s)$ is generated by the spherical vector for any $\nu \in \mathcal{P}$. Indeed, for $\nu \not| \infty$, $\chi_\nu = \mathbb{1}_\nu$, $s = \frac{1}{2}$ and E_ν not a field we have

Proposition 6.2.2 ([18])**.** *For* $\nu \not| \infty$ *and* E_ν *not a field the degenerate principal series representation* $I_{P_E}\left(\mathbb{1}_\nu, \frac{1}{2}\right)$ *has a unique irreducible quotient which is unramified. In particular it is generated by the spherical vector.*

As for the Archimedean places,

Conjecture 6.2.3. *For* $\nu|\infty$, *the degenerate principal series representation* $I_{P_E}\left(\mathbb{1}_\nu, \frac{1}{2}\right)$ *is generated by the spherical vector.*

We point out here that the proof of Theorem 7.1.2 is independent from Conjecture 6.2.3 and can be carried out using only the bounds in Table 6.1. Also, if this conjecture is false, one can still prove the order of the pole at $s = \frac{1}{2}$ when $\chi = \mathbb{1}$ is at most 3 when E is split and 2 when $E = F \times K$ via other methods.

Remark 6.2.4. We do not discuss the cases where $s = \frac{1}{2}$, E is non-split and χ is non-trivial in this thesis, as the analysis of the behavior of the local intertwining operator in these cases is more complicated and not necessary for the purposes of Chapter 7.

Dealing with these cases and the description of the residual representations is an ongoing project.

Remark 6.2.5. In the course of the proof we use Equation (6.5) to evaluate the constant term and check the cancellation of the poles of the various intertwining operators. Note that in the cases where $s = \frac{1}{2}$ and $\chi = \mathbb{1}$ and E is not a field, it happens that the cancellation is of triples or quintuples of intertwining operators.

Proof. The poles at $s=\frac{3}{2}$ and $\chi=\chi_E$ are treated in [16]. Also, since the poles at $s=\frac{5}{2}$ and $\chi=\mathbb{1}$ arise from a single intertwining operator, they cannot be canceled. In what follows we treat the rest of the poles. We leave the discussion of the square integrability of the residual representations to the end of this chapter.

6.2.1 E is a field, $s=\frac{1}{2}$, $\chi=\mathbb{1}$

The intertwining operators in this case have poles at most of order 2. We show that the pole of order 2 cancels and that the pole of order 1 does not.

1. A pole of order 2 is obtained by the intertwining operators associated with $w[212]$ and $w[2121]$. For all such w-s, $w^{-1}\cdot\chi_s(t)=\frac{1}{|t_2|_F}$.

   We note that

   $$M(w_{2121},\chi_s)=M\left(w_1,w_{212}^{-1}\cdot\chi_s\right)\circ M(w_{212},\chi_s).$$

   According to Corollary 6.1.8, we conclude that

   $$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^2 M(w_{2121},\chi_s)=-\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^2 M(w_{212},\chi_s).$$

   and hence

   $$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^2\mathcal{E}_E(\chi,f_s,s,g)_{B_E}=0\qquad\forall f_s\in I_{P_E}(\chi,s).$$

   Thus, the pole of order 2 is canceled.

2. A pole of order 1 is obtained by the intertwining operators associated with $w[21]$ and $w[21212]$. For all such w-s, $w^{-1}\cdot\chi_s(t)=\frac{|t_2|_F}{|t_1|_E}$.

   In this case, the residues of $M_{w_{21}}$ and $M_{w_{21212}}$ are non-zero. Furthermore, $\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M_{w_{21}}f_s|_{T_E}$ and $\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M_{w_{21212}}f_s|_{T_E}$ lie in the representation $w_{21}^{-1}\cdot\chi_s(t)=\frac{|t_2|_F}{|t_1|_E}$ of T_E. This representation is not isomorphic to the representation

$w_{212}^{-1} \cdot \chi_s(t) = \frac{1}{|t_2|_F}$ in which $\lim_{s \to \frac{1}{2}} \left(s - \frac{1}{2}\right)(M_{w_{212}} + M_{w_{2121}}) f_s|_{T_E}$ lies. Therefore, the two different terms cannot cancel each other. Thus it is enough to prove that the pole of $M_{w_{21}}$ and $M_{w_{21212}}$ do not cancel. We show this for the global spherical vector f_s^0.

We write

$$\xi_F(s) = \frac{\gamma_{-1}}{s-1} + \gamma_0 + ..., \quad \xi_K(s) = \frac{\epsilon_{-1}}{s-1} + \epsilon_0 + ...$$

It holds that

$$\begin{aligned}
&\lim_{s \to \frac{1}{2}} \left(s - \frac{1}{2}\right)(M_{w_{21}} + M_{w_{21212}}) f_s^0 \\
&= \lim_{s \to \frac{1}{2}} \left(s - \frac{1}{2}\right) \left(\frac{\xi_F\left(s + \frac{3}{2}\right) \xi_E\left(s + \frac{1}{2}\right)}{\xi_F\left(s + \frac{5}{2}\right) \xi_E\left(s + \frac{3}{2}\right)} + \frac{\xi_F\left(s - \frac{3}{2}\right) \xi_F\left(s + \frac{3}{2}\right) \xi_E\left(s - \frac{1}{2}\right) \xi_F(2s)}{\xi_F\left(s - \frac{1}{2}\right) \xi_F\left(s + \frac{5}{2}\right) \xi_E\left(s + \frac{3}{2}\right) \xi_F(2s+1)} \right) \\
&= \frac{\xi_F(2)}{\xi_F(3)\,\xi_E(2)} \lim_{s \to \frac{1}{2}} \left(s - \frac{1}{2}\right) \left(\xi_F\left(s + \frac{1}{2}\right) + \frac{\xi_F(-1)}{\xi_F(2)} \frac{\xi_E\left(s - \frac{1}{2}\right) \xi_F(2s)}{\xi_F\left(s - \frac{1}{2}\right)} \right) \\
&= \frac{\xi_F(2)}{\xi_F(3)\,\xi_E(2)} \left(-\gamma_{-1} + \frac{\epsilon_{-1} \frac{1}{2} \gamma_{-1}}{-\gamma_{-1}} \right) = \frac{\xi_F(2)}{\xi_F(3)\,\xi_E(2)} \left(\gamma_{-1} + \frac{1}{2} \epsilon_{-1} \right) \neq 0.
\end{aligned}$$

Here we use the fact that γ_{-1} and ϵ_{-1} are positive numbers due to the class number formula [45, p. 37].

6.2.2 E is a field, $s = \frac{1}{2}$, $\chi^2 = \mathbb{1}$, $\chi \neq \mathbb{1}$

A pole of order 1 is obtained by the intertwining operators associated with $w[212]$, $w[2121]$ and $w[21212]$.

We have

$$\begin{aligned}
w_{212}^{-1} \cdot \chi_s(t) &= \chi\left(\mathrm{Nm}_{E/F}(t_1)\right) \frac{1}{|t_2|_F} \\
w_{2121}^{-1} \cdot \chi_s(t) &= \chi\left(t_2 \,\mathrm{Nm}_{E/F}(t_1)\right) \frac{1}{|t_2|_F} \\
w_{21212}^{-1} \cdot \chi_s(t) &= \chi(t_2) \frac{|t_2|_F}{|t_1|_E}.
\end{aligned}$$

In this case, $M_{w_{212}}$, $M_{w_{2121}}$ and $M_{w_{21212}}$ are non-zero. Furthermore $\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M_{w_{212}}f_s|_{T_E}$, $\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M_{w_{2121}}f_s|_{T_E}$ and $\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M_{w_{21212}}f_s|_{T_E}$ lie in three non-isomorphic representations of T_E and therefore

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\mathcal{E}_E\left(\chi,f_s,s,g\right)_{B_E}\not\equiv 0.$$

Hence, the pole of order 1 does not cancel.

6.2.3 E is a field, $s=\frac{3}{2}$, $\chi=\mathbb{1}$

A pole of order 1 is obtained by the intertwining operators associated with $w\left[2121\right]$ and $w\left[21212\right]$. For all such w-s, $w^{-1}\cdot\chi_s\left(t\right)=\frac{1}{|t_1|_E}$.

We note that

$$M\left(w_{21212},\chi_s\right)=M\left(w_2,w_{2121}^{-1}\cdot\chi_s\right)\circ M\left(w_{2121},\chi_s\right).$$

According to Corollary 6.1.8, we conclude that

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{3}{2}\right)M\left(w_{21212},\chi_s\right)=-\lim_{s\to\frac{1}{2}}\left(s-\frac{3}{2}\right)M\left(w_{2121},\chi_s\right).$$

and therefore

$$\lim_{s\to\frac{3}{2}}\left(s-\frac{1}{2}\right)\mathcal{E}_E\left(\mathbb{1},f_s,s,g\right)_{B_E}=0.$$

Hence the pole of order 1 cancels and $\mathcal{E}_E\left(\chi,f_s,s,g\right)$ is holomorphic at $s=\frac{3}{2}$ in this case.

6.2.4 $E=F\times K$, K is a field, $s=\frac{1}{2}$, $\chi=\mathbb{1}$

The non-holomorphic intertwining operators in this case admit poles of order 1, 2 and 3. As we do not know the full Laurent expansion of intertwining operators and we would like to prove the cancellation of more than one order, we proceed differently in this case and also in the split case for $\chi = \mathbb{1}$ at $s = \frac{1}{2}$. We recall from Proposition 6.2.2 and Conjecture 6.2.3 that for any ν the induced representation $\mathrm{Ind}_{B_E}^{H_E} \chi_{s,\nu}$ is of length 2 and the unique irreducible quotient is spherical. Hence, the global induced representation $\mathrm{Ind}_{B_E}^{H_E} \chi_s$ is cyclic and generated by the spherical section. In particular, in order to prove that the pole of $\mathcal{E}_E(\mathbb{1}, f_s, s, g)$ is at most of order 1 it is enough to prove that the pole of $\mathcal{E}_E(\mathbb{1}, f_s^0, s, g)$ is of order 1 where f_s^0 is the unique spherical section such that $f_s^0(1) = 1$. We first prove that the pole of order 3 and the pole of order 2 cancel. Later we prove that the pole of order 1 is obtained.

We write

$$\xi_F(s) = \frac{\gamma_{-1}}{s-1} + \gamma_0 + ..., \quad \xi_K(s) = \frac{\delta_{-1}}{s-1} + \delta_0 + ...$$

The functional equation, Equation (6.6), yields

$$\frac{\xi_F\left(s-\frac{1}{2}\right)}{\xi_F\left(s+\frac{1}{2}\right)} = \frac{\xi_F\left(\frac{3}{2}-s\right)}{\xi_F\left(s+\frac{1}{2}\right)} \underset{s\to\frac{1}{2}}{\longrightarrow} -1 \tag{6.19}$$

and also

$$\gamma_{-1} = \mathrm{Res}\left(\xi_F(s), 1\right) = -\mathrm{Res}\left(\xi_F(s), 0\right), \quad \delta_{-1} = \mathrm{Res}\left(\xi_K(s), 1\right) = -\mathrm{Res}\left(\xi_K(s), 0\right). \tag{6.20}$$

1. The elements $w \in W$ such that $M_w f_s^0$ admits a pole of order 3 are $w[2132]$, $w[21321]$, $w[21323]$ and $w[213213]$. Note that

$$w_{2132}^{-1} \cdot \chi_s(t) = w_{21321}^{-1} \cdot \chi_s(t) = w_{21323}^{-1} \cdot \chi_s(t) = w_{213213}^{-1} \cdot \chi_s(t) = \frac{1}{|t_2|_F}$$

and hence for any w from this list

$$M_w f_s^0(t) = \frac{1}{|t_2|_F} M_w f_s^0(1) \quad \forall t \in T_E(\mathbb{A}).$$

Hence, in order to prove that these elements can contribute at most a simple pole, it is enough to prove that this is the case after evaluating the coefficients of the Laurent series at $t = 1$.

We consider the leading terms in the Laurent series of $M_w f_s^0(1)$ for the corresponding intertwining operators:

$$\frac{\xi_F\left(s+\frac{1}{2}\right)\xi_K\left(s+\frac{1}{2}\right)\xi_F(2s)}{\xi_F\left(s+\frac{5}{2}\right)\xi_K\left(s+\frac{3}{2}\right)\xi_F(2s+1)}+\frac{\xi_F\left(s-\frac{1}{2}\right)\xi_K\left(s+\frac{1}{2}\right)\xi_F(2s)}{\xi_F\left(s+\frac{5}{2}\right)\xi_K\left(s+\frac{3}{2}\right)\xi_F(2s+1)}$$
$$+\frac{\xi_F\left(s+\frac{1}{2}\right)\xi_K\left(s-\frac{1}{2}\right)\xi_F(2s)}{\xi_F\left(s+\frac{5}{2}\right)\xi_K\left(s+\frac{3}{2}\right)\xi_F(2s+1)}+\frac{\xi_F\left(s-\frac{1}{2}\right)\xi_K\left(s-\frac{1}{2}\right)\xi_F(2s)}{\xi_F\left(s+\frac{5}{2}\right)\xi_K\left(s+\frac{3}{2}\right)\xi_F(2s+1)}$$

As the denumerators are equal, holomorphic and non-vanishing at $s=\frac{1}{2}$ it is enough to prove that the sum of the numerators admit at most a simple pole. Indeed,

$$\xi_F\left(s+\frac{1}{2}\right)\xi_K\left(s+\frac{1}{2}\right)\xi_F(2s)+\xi_F\left(s-\frac{1}{2}\right)\xi_K\left(s+\frac{1}{2}\right)\xi_F(2s)$$
$$+\xi_F\left(s+\frac{1}{2}\right)\xi_K\left(s-\frac{1}{2}\right)\xi_F(2s)+\xi_F\left(s-\frac{1}{2}\right)\xi_K\left(s-\frac{1}{2}\right)\xi_F(2s)$$
$$=\frac{\gamma_{-1}^2\delta_{-1}-\gamma_{-1}^2\delta_{-1}-\gamma_{-1}^2\delta_{-1}+\gamma_{-1}^2\delta_{-1}}{2\left(s-\frac{1}{2}\right)^3}$$
$$+\frac{\gamma_{-1}\left(3\delta_{-1}\gamma_0+\gamma_{-1}\delta_0\right)-\gamma_{-1}\left(\gamma_{-1}\delta_0+\delta_{-1}\gamma_0\right)\gamma_{-1}\left(\gamma_{-1}\delta_0-3\delta_{-1}\gamma_0\right)+\gamma_{-1}\left(\delta_{-1}\gamma_0-\gamma_{-1}\delta_0\right)}{2\left(s-\frac{1}{2}\right)^2}$$
$$+o\left(\left(s-\frac{1}{2}\right)^{-2}\right)=o\left(\left(s-\frac{1}{2}\right)^{-2}\right).$$

In particular,

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^3\mathcal{E}_E\left(\mathbb{1},f_s^0,s,g\right)_{B_E}=0.$$

2. The elements $w\in W$ such that $M_w f_s^0$ admits a pole of order 2 are $w[213]$, $w[2321]$ and $w[2132132]$.

   Note that

$$w_{213}^{-1}\cdot\chi_s(t)=w_{2321}^{-1}\cdot\chi_s(t)=w_{2132132}^{-1}\cdot\chi_s(t)=\frac{|t_2|_F}{|t_1|_F\,|t_3|_K}$$

   and hence for any w from this list

$$M_w f_s^0(t)=\frac{|t_2|_F}{|t_1|_F\,|t_3|_K}M_w f_s^0(1)\quad\forall t\in T_E(\mathbb{A}).$$

Hence, in order to prove that these elements can contribute at most a simple pole, it is enough to prove that this is the case after evaluating the coefficients of the Laurent series at $t = 1$.

We consider the leading terms in the Laurent series of $M_w f_s^0(1)$ for the corresponding intertwining operators:

$$\frac{\xi_F\left(s+\frac{1}{2}\right)\xi_K\left(s+\frac{1}{2}\right)}{\xi_F\left(s+\frac{5}{2}\right)\xi_K\left(s+\frac{3}{2}\right)}+\frac{\xi_F\left(s+\frac{3}{2}\right)\xi_K\left(s+\frac{1}{2}\right)\xi_F\left(s-\frac{1}{2}\right)\xi_F(2s)}{\xi_F\left(s+\frac{5}{2}\right)\xi_K\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{1}{2}\right)\xi_F(2s+1)}$$
$$+\frac{\xi_F\left(s-\frac{3}{2}\right)\xi_K\left(s-\frac{1}{2}\right)\xi_F(2s)}{\xi_F\left(s+\frac{5}{2}\right)\xi_K\left(s+\frac{3}{2}\right)\xi_F(2s+1)}$$
$$=\frac{\xi_F(2)\gamma_{-1}\delta_{-1}-\frac{1}{2}\xi_F(2)\gamma_{-1}\delta_{-1}-\frac{1}{2}\xi_F(-1)\gamma_{-1}\delta_{-1}}{\xi_F(2)\xi_K(2)\xi_F(3)\left(s-\frac{1}{2}\right)^2}+o\left(\left(s-\frac{1}{2}\right)^{-2}\right)$$
$$=o\left(\left(s-\frac{1}{2}\right)^{-2}\right).$$

In particular,

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^2\mathcal{E}_E\left(1,f_s^0,s,g\right)_{B_E}=0.$$

3. The elements $w\in W$ such that $M_w f_s^0$ that might contribute to the pole of order 1 are $w[21]$, $w[23]$, $w[213]$, $w[232]$, $w[2132]$, $w[2321]$, $w[21321]$, $w[21323]$, $w[213213]$ and $w[2132132]$. Here we note that for $\chi=1$, $s=\frac{1}{2}$ and $t=(t_1,t_2,t_3)$ we have

$$w_{21}^{-1}\cdot\chi_s(t)=\frac{|t_3|_K}{|t_1|_F\,|t_2|_F},$$
$$w_{23}^{-1}\cdot\chi_s(t)=w_{232}^{-1}\cdot\chi_s(t)=\frac{|t_1|_F}{|t_3|_K},$$
$$w_{213}^{-1}\cdot\chi_s(t)=w_{2321}^{-1}\cdot\chi_s(t)=w_{2132132}^{-1}\cdot\chi_s(t)=\frac{|t_2|_F}{|t_1|_F\,|t_3|_K},$$
$$w_{2132}^{-1}\cdot\chi_s(t)=w_{21321}^{-1}\cdot\chi_s(t)=w_{21323}^{-1}\cdot\chi_s(t)=w_{213213}^{-1}\cdot\chi_s(t)=\frac{1}{|t_2|_F},$$

and thus, $M_{w[21]}$ indeed admits a pole of order 1 at $s=\frac{1}{2}$ and this pole cannot be canceled by the poles of other intertwining operators, as their image lies in different representations.

In conclusion, $\mathcal{E}_E\left(\mathbb{1},f_s^0,s,g\right)$ admits a pole of order 1 at $s=\frac{1}{2}$.

6.2.5 $E = F \times K$, K is a field, $s = \frac{3}{2}$, $\chi = \mathbb{1}$

The intertwining operators in this case have poles at most of order 2. We show that the pole of order 2 cancels and that the pole of order 1 does not.

1. A pole of order 2 is attained by the intertwining operators associated with $w[213213]$ and $w[2132132]$. For all such w-s, $w^{-1} \cdot \chi_s(t) = \frac{1}{|t_1|_F |t_3|_K}$.

   We note that

   $$M(w_{2132132}, \chi_s) = M\left(w_2, w_{213213}^{-1} \cdot \chi_s\right) \circ M(w_{213213}, \chi_s).$$

   According to Corollary 6.1.8, we conclude that

   $$\lim_{s \to \frac{3}{2}} \left(s - \frac{3}{2}\right)^2 M(w_{2132132}, \chi_s) = -\lim_{s \to \frac{3}{2}} \left(s - \frac{3}{2}\right)^2 M(w_{213213}, \chi_s)$$

   and hence

   $$\lim_{s \to \frac{3}{2}} \left(s - \frac{3}{2}\right)^2 \mathcal{E}_E(\mathbb{1}, f_s, s, g)_{B_E} = 0.$$

2. A pole of order 1 is obtained by the intertwining operators associated with $w[232]$, $w[2321]$, $w[21321]$ and $w[21323]$. We have

   $$\begin{aligned} w_{232}^{-1} \cdot \chi_s(t) &= \frac{|t_1|_F^3}{|t_2|_F |t_3|_K} \\ w_{2321}^{-1} \cdot \chi_s(t) &= \frac{|t_2|_F^2}{|t_1|_F^3 |t_3|_K} \\ w_{21321}^{-1} \cdot \chi_s(t) &= \frac{|t_3|_K}{|t_1 t_2^2|_F} \\ w_{21323}^{-1} \cdot \chi_s(t) &= \frac{|t_1|_F}{|t_2|_F |t_3|_K}. \end{aligned}$$

   In this case, $M_{w_{232}}$, $M_{w_{2321}}$, $M_{w_{21321}}$ and $M_{w_{21323}}$ are non-zero. Furthermore $\lim_{s \to \frac{3}{2}} \left(s - \frac{3}{2}\right) M_{w_{232}} f_s|_{T_E}$, $\lim_{s \to \frac{3}{2}} \left(s - \frac{3}{2}\right) M_{w_{2321}} f_s|_{T_E}$, $\lim_{s \to \frac{3}{2}} \left(s - \frac{3}{2}\right) M_{w_{21321}} f_s|_{T_E}$ and

$\lim\limits_{s\to\frac{3}{2}}\left(s-\frac{3}{2}\right)M_{w_{21323}}f_s|_{T_E}$ lie in four non-isomorphic representations of T_E, which in turn are not isomorphic to the spaces in the previous item, and therefore

$$\lim_{s\to\frac{3}{2}}\left(s-\frac{3}{2}\right)\mathcal{E}_E\left(\mathbb{1},f_s,s,g\right)_{B_E}\not\equiv 0.$$

Since the pole of order 2 cancels and the pole of order 1 does not, then the order of the pole in this case is 1.

6.2.6 $E=F\times F\times F$, $s=\frac{1}{2}$, $\chi=\mathbb{1}$

The poles of the intertwining operators in this case admit poles of order 1, 2, 3 and 4. As we do not know the full Laurent expansion of intertwining operators and we would like to prove cancellation of more than one order, we proceed in a way similar to the case $E=F\times K$ for $\chi=\mathbb{1}$ at $s=\frac{1}{2}$. We recall from Proposition 6.2.2 and Conjecture 6.2.3 that, for any ν, the induced representation $\mathrm{Ind}_{B_E}^{H_E}\chi_{s,\nu}$ is of length 2 and the unique irreducible quotient is spherical. Hence the global induced representation $\mathrm{Ind}_{B_E}^{H_E}\chi_s$ is cyclic and generated by the spherical section. In particular, in order to prove that the pole of $\mathcal{E}_E\left(\mathbb{1},f_s,s,g\right)$ is of order 1 it is enough to prove that the pole of $\mathcal{E}_E\left(\mathbb{1},f_s^0,s,g\right)$ is of order 1 where f_s^0 is the unique spherical section such that $f_s^0\left(1\right)=1$. We first prove that the pole of order 4, the pole of order 3 and the pole of order 2 cancel. Then we prove that the pole of order 1 is obtained.

We write

$$\xi\left(s\right)=\frac{\gamma_{-1}}{s-1}+\gamma_0+\gamma_1\left(s-1\right)...$$

The functional equation, Equation (6.6), yields

$$\xi\left(s\right)=\xi\left(1-s\right)=-\frac{\gamma_{-1}}{s}+\gamma_0-\gamma_1-... \tag{6.21}$$

and also

$$\gamma_{-1}=\mathrm{Res}\left(\xi\left(s\right),1\right)=-\,\mathrm{Res}\left(\xi\left(s\right),0\right). \tag{6.22}$$

1. The elements $w\in W$ such that $M_wf_s^0$ that might contribute to the pole of order 4 are $w[21342]$, $w[213421]$, $w[213423]$, $w[213424]$, $w[2134213]$, $w[2134214]$, $w[2134234]$ and $w[21342134]$.

Note that

$$w_{21342}^{-1}\cdot\chi_s(t)=w_{213421}^{-1}\cdot\chi_s(t)=w_{213423}^{-1}\cdot\chi_s(t)=w_{213424}^{-1}\cdot\chi_s(t)$$
$$=w_{2134213}^{-1}\cdot\chi_s(t)=w_{2134214}^{-1}\cdot\chi_s(t)=w_{2134234}^{-1}\cdot\chi_s(t)=w_{21342134}^{-1}\cdot\chi_s(t)=\left|\frac{1}{t_2}\right|.$$

and hence for any w from this list

$$M_w f_s^0(t)=\frac{1}{|t_2|_F}M_w f_s^0(1)\quad\forall t\in T_E(\mathbb{A}).$$

Hence, in order to prove that these elements can contribute at most a simple pole, it is enough to prove that this is so after evaluating the coefficients of the Laurent series at $t=1$.

We consider the leading terms in the Laurent series of $M_w f_s^0(1)$ for the corresponding intertwining operators:

$$\frac{\xi\left(s+\frac{1}{2}\right)^3\xi(2s)}{\xi\left(s+\frac{3}{2}\right)^2\xi\left(s+\frac{5}{2}\right)\xi(2s+1)}+3\frac{\xi\left(s-\frac{1}{2}\right)\xi\left(s+\frac{1}{2}\right)^2\xi(2s)}{\xi\left(s+\frac{3}{2}\right)^2\xi\left(s+\frac{5}{2}\right)\xi(2s+1)}$$
$$+3\frac{\xi\left(s-\frac{1}{2}\right)^2\xi\left(s+\frac{1}{2}\right)\xi(2s)}{\xi\left(s+\frac{3}{2}\right)^2\xi\left(s+\frac{5}{2}\right)\xi(2s+1)}+\frac{\xi\left(s-\frac{1}{2}\right)^3\xi(2s)}{\xi\left(s+\frac{3}{2}\right)^2\xi\left(s+\frac{5}{2}\right)\xi(2s+1)}.$$

As the denumerators are equal, holomorphic and non-vanishing at $s=\frac{1}{2}$, it is enough to prove that the sum of the numerators admits at most a simple pole. Indeed,

$$\xi\left(s+\frac{1}{2}\right)^3\xi(2s)+3\,\xi\left(s-\frac{1}{2}\right)\xi\left(s+\frac{1}{2}\right)^2\xi(2s)$$
$$+3\,\xi\left(s-\frac{1}{2}\right)^2\xi\left(s+\frac{1}{2}\right)\xi(2s)+\xi\left(s-\frac{1}{2}\right)^3\xi(2s)$$
$$=\frac{\gamma_{-1}^4-3\gamma_{-1}^4+3\gamma_{-1}^4-\gamma_{-1}^4}{2\left(s-\frac{1}{2}\right)^4}+\frac{5\gamma_{-1}^3\gamma_0-9\gamma_{-1}^3\gamma_0+3\gamma_{-1}^3\gamma_0+\gamma_{-1}^3\gamma_0}{2\left(s-\frac{1}{2}\right)^3}$$
$$+\frac{\left(7\gamma_{-1}^3\gamma_1+9\gamma_{-1}^2\gamma_0\right)+3\left(-7\gamma_{-1}^3\gamma_1-\gamma_{-1}^2\gamma_0\right)+3\left(7\gamma_{-1}^3\gamma_1-3\gamma_{-1}^2\gamma_0\right)+\left(-7\gamma_{-1}^3\gamma_1+3\gamma_{-1}^2\gamma_0\right)}{2\left(s-\frac{1}{2}\right)^2}$$
$$+o\left(\left(s-\frac{1}{2}\right)^{-2}\right)=o\left(\left(s-\frac{1}{2}\right)^{-2}\right).$$

In particular,

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^4\mathcal{E}_E\left(\mathbb{1},f_s^0,s,g\right)_{B_E}=0.$$

2. The elements $w\in W$ such that $M_wf_s^0$ that might contribute to the pole of order 3 are $w[2134]$, $w[21324]$, $w[21423]$, $w[23421]$ and $w[213421342]$.

   Note that

   $$w_{2134}^{-1}\cdot\chi_s(t)=w_{23421}^{-1}\cdot\chi_s(t)=w_{21423}^{-1}\cdot\chi_s(t)=w_{21324}^{-1}\cdot\chi_s(t)=w_{213421342}^{-1}\cdot\chi_s(t)=\left|\frac{t_2}{t_1t_3t_4}\right|$$

   and hence for any w from this list

   $$M_wf_s^0(t)=\left|\frac{t_2}{t_1t_3t_4}\right|M_wf_s^0(1)\quad\forall t\in T_E(\mathbb{A}).$$

   Hence, in order to prove that these elements can contribute at most a simple pole, it is enough to prove that this is so after evaluating the coefficients of the Laurent series at $t=1$.

   We consider the leading terms in the Laurent series of $M_wf_s^0(1)$ for the corresponding intertwining operators:

   $$\frac{\xi\left(s+\frac{1}{2}\right)^3}{\xi\left(s+\frac{3}{2}\right)^2\xi\left(s+\frac{5}{2}\right)}+3\frac{\xi\left(s-\frac{1}{2}\right)\xi\left(s+\frac{1}{2}\right)\xi(2s)}{\xi\left(s+\frac{3}{2}\right)\xi\left(s+\frac{5}{2}\right)\xi(2s+1)}+\frac{\xi\left(s-\frac{1}{2}\right)^2\xi\left(s-\frac{3}{2}\right)\xi(2s)}{\xi\left(s+\frac{5}{2}\right)\xi\left(s+\frac{3}{2}\right)^2\xi(2s+1)}$$
   $$=\frac{\xi\left(s+\frac{1}{2}\right)^3\xi(2s+1)+3\,\xi\left(s-\frac{1}{2}\right)\xi\left(s+\frac{1}{2}\right)\xi(2s)\,\xi\left(s+\frac{3}{2}\right)+\xi\left(s-\frac{1}{2}\right)^2\xi\left(s-\frac{3}{2}\right)\xi(2s)}{\xi\left(s+\frac{5}{2}\right)\xi\left(s+\frac{3}{2}\right)^2\xi(2s+1)}.$$

   It is enough to prove that the numerator admits at most a simple pole. Indeed,

   $$\begin{aligned}&\xi\left(s+\frac{1}{2}\right)^3\xi(2s+1)+3\,\xi\left(s-\frac{1}{2}\right)\xi\left(s+\frac{1}{2}\right)\xi(2s)\,\xi\left(s+\frac{3}{2}\right)\\&+\xi\left(s-\frac{1}{2}\right)^2\xi\left(s-\frac{3}{2}\right)\xi(2s)\\&=\frac{\xi(2)\,\gamma_{-1}^3-\frac{3}{2}\,\xi(2)\,\gamma_{-1}^3+\frac{1}{2}\,\xi(-1)\,\gamma_{-1}^3}{\left(s-\frac{1}{2}\right)^3}\\&+\frac{\left(3a_0\gamma_{-1}^2\gamma_0+2a_1\gamma_{-1}^3\right)-3\left(a_0\gamma_{-1}^2\gamma_0+\frac{1}{2}\gamma_{-1}^3\right)-\frac{1}{2}a_1\gamma_{-1}^3}{\left(s-\frac{1}{2}\right)^2}\\&+o\left(\left(s-\frac{1}{2}\right)^{-2}\right)=o\left(\left(s-\frac{1}{2}\right)^{-2}\right).\end{aligned}$$

Here

$$\xi\left(2+\epsilon\right)=a_0+a_1\epsilon+o\left(\epsilon\right),\quad a_0=\xi\left(2\right).$$

3. The elements $w\in W$ such that $M_wf_s^0$ that might contribute to the pole of order 2 are $w[213]$, $w[214]$, $w[234]$, $w[2132]$, $w[2142]$ and $w[2342]$.

   Note that

$$\begin{aligned}
w_{213}^{-1}\cdot\chi_s\left(t\right)=w_{2132}^{-1}\cdot\chi_s\left(t\right)&=\left|\frac{t_4}{t_1t_3}\right|\\
w_{214}^{-1}\cdot\chi_s\left(t\right)=w_{2142}^{-1}\cdot\chi_s\left(t\right)&=\left|\frac{t_3}{t_1t_4}\right|\\
w_{234}^{-1}\cdot\chi_s\left(t\right)=w_{2342}^{-1}\cdot\chi_s\left(t\right)&=\left|\frac{t_1}{t_3t_4}\right|.
\end{aligned}$$

   These items cancel in pairs. In particular, considering the leading terms in the Laurent series of $M_wf_s^0\left(1\right)$ for the corresponding intertwining operators,

$$3\frac{\xi\left(s+\frac{1}{2}\right)^2}{\xi\left(s+\frac{3}{2}\right)\xi\left(s+\frac{5}{2}\right)}+3\frac{\xi\left(s+\frac{1}{2}\right)\xi\left(s-\frac{1}{2}\right)}{\xi\left(s+\frac{5}{2}\right)\xi\left(s+\frac{3}{2}\right)}$$

   As the denumerators are equal, holomorphic and non-vanishing at $s=\frac{1}{2}$, it is enough to prove that the sum of the numerators admits at most a simple pole. Indeed,

$$\begin{aligned}
&\xi\left(s+\frac{1}{2}\right)^2+\xi\left(s+\frac{1}{2}\right)\xi\left(s-\frac{1}{2}\right)\\
&=\frac{\xi\left(2\right)^2\gamma_{-1}^2-\xi\left(2\right)^2\gamma_{-1}^2}{\left(s-\frac{1}{2}\right)^2}+o\left(\left(s-\frac{1}{2}\right)^{-2}\right)=o\left(\left(s-\frac{1}{2}\right)^{-2}\right).
\end{aligned}$$

4. The elements $w\in W$ such that $M_wf_s^0$ that might contribute to the pole of order 1 are $w[21]$, $w[23]$, $w[24]$, $w[213]$, $w[214]$, $w[234]$, $w[2132]$, $w[2142]$, $w[2342]$, $w[2134]$, $w[21324]$, $w[21423]$, $w[23421]$, $w[21342]$, $w[213421]$, $w[213423]$, $w[213424]$, $w[2134213]$, $w[2134214]$, $w[2134234]$, $w[21342134]$ and $w[213421342]$. We note that for $\chi=1$,

$s=\frac{1}{2}$ and $t=(t_1,t_2,t_3,t_4)$ we have

$$w_{21}^{-1}\cdot\chi_s(t)=\left|\frac{t_3t_4}{t_1t_2}\right|,\quad w_{23}^{-1}\cdot\chi_s(t)=\left|\frac{t_1t_4}{t_2t_3}\right|,\quad w_{24}^{-1}\cdot\chi_s(t)=\left|\frac{t_1t_3}{t_2t_4}\right|$$

$$w_{213}^{-1}\cdot\chi_s(t)=w_{2132}^{-1}\cdot\chi_s(t)=\left|\frac{t_4}{t_1t_3}\right|$$

$$w_{214}^{-1}\cdot\chi_s(t)=w_{2142}^{-1}\cdot\chi_s(t)=\left|\frac{t_3}{t_1t_4}\right|$$

$$w_{234}^{-1}\cdot\chi_s(t)=w_{2342}^{-1}\cdot\chi_s(t)=\left|\frac{t_1}{t_3t_4}\right|$$

$$w_{2134}^{-1}\cdot\chi_s(t)=w_{23421}^{-1}\cdot\chi_s(t)=w_{21423}^{-1}\cdot\chi_s(t)=w_{21324}^{-1}\cdot\chi_s(t)=w_{213421342}^{-1}\cdot\chi_s(t)=\left|\frac{t_2}{t_1t_3t_4}\right|$$

$$w_{21342}^{-1}\cdot\chi_s(t)=w_{213421}^{-1}\cdot\chi_s(t)=w_{213423}^{-1}\cdot\chi_s(t)=w_{213424}^{-1}\cdot\chi_s(t)$$

$$=w_{2134213}^{-1}\cdot\chi_s(t)=w_{2134214}^{-1}\cdot\chi_s(t)=w_{2134234}^{-1}\cdot\chi_s(t)=w_{21342134}^{-1}\cdot\chi_s(t)=\left|\frac{1}{t_2}\right|.$$

And so, $M_{w[21]}$, $M_{w[23]}$ and $M_{w[24]}$ indeed admit a pole of order 1 at $s=\frac{1}{2}$ and this pole cannot be canceled by the poles of other intertwining operators since their image lies in different representations.

In conclusion, $\mathcal{E}_E\left(\mathbb{1},f_s^0,s,g\right)$ admits a pole of order 1 at $s=\frac{1}{2}$.

6.2.7 $E=F\times F\times F$, $s=\frac{1}{2}$, $\chi^2=\mathbb{1}$, $\chi\neq\mathbb{1}$

A pole of order 1 is obtained by the intertwining operators associated with $w[21324]$, $w[21423]$, $w[23421]$, $w[21342]$, $w[213421]$, $w[213423]$, $w[213424]$, $w[2134213]$, $w[2134214]$, $w[2134234]$, $w[21342134]$ and $w[213421342]$. We have

$$w_{21324}^{-1}\cdot\chi_s(t)=w_{21423}^{-1}\cdot\chi_s(t)=w_{23421}^{-1}\cdot\chi_s(t)=w_{213421342}^{-1}\cdot\chi_s(t)=\chi(t_2)\left|\frac{t_2}{t_1t_3t_4}\right|$$

$$w_{21342}^{-1}\cdot\chi_s(t)=w_{2134213}^{-1}\cdot\chi_s(t)=w_{2134214}^{-1}\cdot\chi_s(t)=w_{2134234}^{-1}\cdot\chi_s(t)=\chi(t_1t_3t_4)\left|\frac{1}{t_2}\right|$$

$$w_{213421}^{-1}\cdot\chi_s(t)=w_{213423}^{-1}\cdot\chi_s(t)=w_{213424}^{-1}\cdot\chi_s(t)=w_{21342134}^{-1}\cdot\chi_s(t)=\chi(t_1t_2t_3t_4)\left|\frac{1}{t_2}\right|.$$

We note that

$$\begin{aligned}
M\left(w_{21342134},\chi_s\right) &= M\left(w_{13},w_{213424}^{-1}\cdot\chi_s\right)\circ M\left(w_{213424},\chi_s\right)\\
M\left(w_{21342134},\chi_s\right) &= M\left(w_{14},w_{213423}^{-1}\cdot\chi_s\right)\circ M\left(w_{213423},\chi_s\right)\\
M\left(w_{21342134},\chi_s\right) &= M\left(w_{34},w_{213421}^{-1}\cdot\chi_s\right)\circ M\left(w_{213421},\chi_s\right)\\
M\left(w_{2134213},\chi_s\right) &= M\left(w_{13},w_{21342}^{-1}\cdot\chi_s\right)\circ M\left(w_{21342},\chi_s\right)\\
M\left(w_{2134214},\chi_s\right) &= M\left(w_{14},w_{21342}^{-1}\cdot\chi_s\right)\circ M\left(w_{21342},\chi_s\right)\\
M\left(w_{2134234},\chi_s\right) &= M\left(w_{34},w_{21342}^{-1}\cdot\chi_s\right)\circ M\left(w_{21342},\chi_s\right).
\end{aligned}$$

According to Corollary 6.1.8, we conclude that

$$\begin{aligned}
\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{21342},\chi_s\right) &= \lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{2134213},\chi_s\right)\\
&= \lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{2134214},\chi_s\right) = \lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{2134234},\chi_s\right),\\
\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{213421},\chi_s\right) &= \lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{213423},\chi_s\right)\\
&= \lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{213424},\chi_s\right) = \lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{21342134},\chi_s\right).
\end{aligned}$$

We do not treat the rest of the terms as they have different exponents; in particular, the pole of order 1 does not cancel.

6.2.8 Square Integrability of the Residual Representations

We now determine for a point where $\mathcal{E}_E\left(\chi,f_s,s,g\right)$ admits a pole, whether the residual representation is square-integrable or not. Before doing so, we recall the following criterion from [32, p. 104].

Lemma 6.2.6 (Langlands' Criterion for Square Integrability)**.** *Assume $\mathcal{E}_E\left(\chi,f_s,s,g\right)$ admits a pole of order n at s_0. We denote by $W_{(\chi,s_o)}$ the subset of all $w\in W$ such that $M\left(w,\chi_s\right)$ admits a pole of order n at $s=s_0$ whose residue is not canceled by other $w\in W$. The residual representation* $\operatorname{Res}_{s=s_0}\mathcal{E}_E\left(\chi,f_s,s,g\right)$ *is a square-integrable representation if and only if $\mathfrak{Re}\left(w^{-1}\cdot\chi_s\left(t\right)\right)<0$ for all $w\in W_{(\chi,s_o)}$.*

Corollary 6.2.7. *Assume that* $\mathcal{E}_E(\chi, f_s, s, g)$ *admits a pole at* $(E, \chi, s_0) \neq \left(F \times K, 1, \frac{3}{2}\right)$ *with* $\mathfrak{Re}(s_0) > 0$*, then* $\operatorname{Res}_{s=s_0} \mathcal{E}_E(\chi, f_s, s, g)$ *is a square-integrable representation of* H_E*.*

It follows from the proof, from Langlands' criterion to square integrability and from the information in Table B.12, Table B.8 and Table B.4 that the residual representation is square-integrable at the points listed in the statement of the theorem. It also follows that it is not square-integrable in the following cases:

- $E = F \times F \times F$ at $s = \frac{1}{2}$ with $\chi = \mathbb{1}$.
- $E = F \times K$ at $s = \frac{1}{2}$ and $s = \frac{3}{2}$ with $\chi = \mathbb{1}$.

The behavior of the residual representation when $E = F \times K$ at $s = \frac{1}{2}$ with $\chi = \chi_K$ requires further study of the local intertwining operators.

□

Chapter 7

Applications

In this chapter we apply Theorem 2.0.2 and Theorem 6.2.1 to the study of the θ-lift from the finite type group S_E to G_2.

It will be useful to note the following corollary to Theorem 2.0.2.

Corollary 7.0.1. *For any $s_0 \in \mathbb{C}$ it holds that*

$$ord_{s=s_0}\left(\mathcal{E}_E\left(\chi, f_s, s, g\right)\right) \geq ord_{s=s_0}\left(\mathcal{L}^S\left(s+\frac{1}{2}, \pi, \chi, \mathfrak{st}\right)\right). \tag{7.1}$$

Remark 7.0.2. Note that $j_E^S(s,\chi)$ is holomorphic for

$$at \tag{7.2}$$

$\mathfrak{Re}(s) > 0$.

As a first application, we resolve a conjecture of J. Hundley and D. Ginzburg ([20]).

Conjecture 7.0.3. *The twisted partial standard $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ can have at most a double pole at $\mathfrak{Re}(s) > 0$.*

Proof. This follows immediately from Corollary 7.0.1 and Theorem 6.2.1. □

7.1 CAP Representations and the Dual Reductive Pair (G_2, S_E)

We would now like to use the results from previous chapters to prove that the **CAP** representations of $G_2(\mathbb{A})$ supporting the Ψ_E-Fourier coefficient are precisely the representations π such that $\mathcal{L}(s,\pi,\chi_E,\mathfrak{st})$ admits a pole at $\frac{3}{2}$ of maximal order. In turn, we prove that these representations are precisely the representations in the image of the θ-lift from S_E. We start by recalling the relevant definitions.

For any étale cubic algebra E over F, the choice of Chevalley-Steinberg system as in Chapter 1 defines a splitting of the exact sequence

$$\{1\} \to H_E^{ad} \to \mathrm{Aut}(H_E) \to S_E \to \{1\}.$$

We can then form the semidirect product $H_E \rtimes S_E$. The centralizer of S_E in H_E is isomorphic to G. This gives rise to a dual reductive pair

$$G \times S_E \hookrightarrow H_E \rtimes S_E.$$

We denote the associated θ-lift from G_2 to S_E by θ_{S_E}. The θ-lift in the other direction is denoted by θ_E. We also denote

$$n_E = \begin{cases} 2, & E = F\times F\times F \\ 1, & \text{otherwise} \end{cases}.$$

We now recall the definition of a **CAP** representation.

Definition 7.1.1. Let $P = M\cdot N \subset G$ be a parabolic subgroup, σ be a cuspidal unitary representation of the Levi part M and χ be a character of M. A cuspidal representation π of $G(\mathbb{A})$ is called a **CAP** (cuspidal attached to parabolic) with respect to P, σ and χ if π is nearly equivalent to a subquotient of $\mathrm{Ind}_P^G \sigma\otimes\chi$.

CAP representations for G_2 were constructed in [16] for the Borel subgroup, in [39] for the Heisenberg parabolic subgroup P and in [15] for the non-Heisenberg maximal parabolic subgroup. Using Corollary 7.0.1 and Table 6.2 we plan to prove that [16] exhausts the list of **CAP** representations with respect to the Borel subgroup.

Theorem 7.1.2. *Let π be a cuspidal representation of $G(\mathbb{A})$ and let E be an étale cubic algebra E over F which is not a non-Galois field extension. The following are equivalent:*

1. *π is a **CAP** representation with respect to B supporting the (U,Ψ_E)-coefficient.*

2. *The partial* $\mathcal{L}$*-function* $\mathcal{L}^S(s,\pi,\chi_E,\mathfrak{st})$ *has a pole of order* n_E *at* $s=2$

3. $\Theta_{S_E}(\pi)\neq 0$*. In particular* π *is nearly equivalent to* $\Theta_E(\mathbf{1}_{S_E})$*, where* $\mathbf{1}_{S_E}$ *here is the automorphic trivial representation of* $S_E(\mathbb{A})$*.*

Proof. The fact that *3* implies *1* and *2* was proven in [16]. We prove that in turn *1* and *2* imply *3*.

2* implies *3 Recall [16, Proposition 5.1].

Proposition 7.1.3. *For any standard section* f_s*, the Eisenstein series* $\mathcal{E}(\chi_E, f_s, s, g)$ *has at most a pole of order* n_E *at* $s=\frac{3}{2}$*. A pole of order* n_E *is attained. Moreover, the space of automorphic forms,*

$$Span_{\mathbb{C}}\left\{\left(s-\frac{3}{2}\right)^{n_E}\mathcal{E}^*_E(\chi_E,s,f_s,g)\Big|_{s=\frac{3}{2}}\right\},$$

is an irreducible square-integrable automorphic representation isomorphic to the minimal representation Π_E *of* H_E*.*

We note that from Theorem 6.2.1 and Corollary 7.0.1 it follows that if $\mathcal{L}^S(s,\pi,\chi_E,\mathfrak{st})$ admits a pole at $s=\frac{3}{2}$ of order n_E then π supports the Ψ_E-Fourier coefficient. Indeed, assume that π does not support the Ψ_E-Fourier coefficient, by the results of [16] π supprts the $\Psi_{E'}$-Fourier coefficient for some étale cubic algebra $E'\neq E$. This,together with Corollary 7.0.1, implies that $\mathcal{E}^*_{E'}(\chi_E,s,f_s,g)$ admits a pole of order n_E at $s=\frac{3}{2}$ contradicting Theorem 6.2.1.

Assume that $\mathcal{L}^S(s,\pi,\chi_E,\mathfrak{st})$ admits a pole of order n_E. According to Corollary 7.0.1, this implies that there exist f and φ such that $\mathcal{Z}_E(\chi,s,\varphi,f)$ admits a pole of order n_E. Taking the residue at $s=\frac{3}{2}$ in Equation (2.1) implies the assertion.

1* implies *3 Let π be a **CAP** representation with respect to B that supports the Ψ_E-Fourier coefficient. We prove that *3* holds by proving that π is nearly equivalent to $\Theta_{H_E}(\mathbf{1}_{S_E})$ where $\mathbf{1}_{S_E}$ is the trivial representation of $S_E(\mathbb{A})$.

Remark 7.1.4. Note that all irreducible automorphic representations of $S_E(\mathbb{A})$ are nearly equivalent to 1_{S_E}.

By the assumption, there exists an automorphic character μ such that π is nearly equivalent to a subquotient of $\operatorname{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\mu$, where the induction here is unitary. Let $\mu\left(h_{2\alpha+\beta}(a)\,h_{3\alpha+2\beta}(b)\right)=\mu_1(a)\,\mu_2(b)$. We denote by $\mu_i(x)=\eta_i(x)\,|x|^{z_i}$, where η_i are unitary characters and $z_i\in\mathbb{R}$. By choosing a Weyl chamber we may assume that

$$0\le z_2\le z_1\le 2z_2. \tag{7.3}$$

According to [16] we need to show:

- If $E=F\times F\times F$ then $\mu_1(t)=\mu_2(t)=|t|$ for any $t\in\mathbb{A}^\times$.
- If $E=F\times K$ then $\mu_1(t)=|t|$ and $\mu_2(t)=\chi_K(t)\,|t|$ for any $t\in\mathbb{A}^\times$, or vice versa.
- If E/F is a cubic Galois extension, then $\mu_1(t)=\mu_2(t)=\chi_E(t)\,|t|$ for any $t\in\mathbb{A}^\times$.

It holds that

$$\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})=\mathcal{L}_F^S(\mu_1\chi,s)\,\mathcal{L}_F^S\left(\mu_1^{-1}\chi,s\right)\mathcal{L}_F^S(\mu_2\chi,s)\,\mathcal{L}_F^S\left(\mu_2^{-1}\chi,s\right)\mathcal{L}_F^S\left(\frac{\mu_1}{\mu_2}\chi,s\right)\mathcal{L}_F^S\left(\frac{\mu_2}{\mu_1}\chi,s\right)\mathcal{L}_F^S(\chi,s).$$

For $\chi(t)=\mu_1(t)\,|t|^{-1}$, $\mathcal{L}^S\left(s,\pi,\mu_1|\cdot|^{-1},\mathfrak{st}\right)$ admits a pole at $s=2$ and hence $\mathcal{E}_E\left(\mu_1|\cdot|^{-1},f_s,s,g\right)$ admits a pole at $s=\frac{3}{2}$. Similarly, $\mathcal{E}_E\left(\mu_2|\cdot|^{-1},f_s,s,g\right)$ also admits a pole at $s=\frac{3}{2}$.

We continue by considering different kinds of E.

- <u>$E=F\times F\times F$:</u> Since $\mathcal{E}_E\left(\mu_1|\cdot|^{-1},f_s,s,g\right)$ and $\mathcal{E}_E\left(\mu_2|\cdot|^{-1},f_s,s,g\right)$ admits a pole at $s=\frac{3}{2}$, it holds that

  $$(z_1,\eta_1),(z_2,\eta_2)\in\left\{(0,\eta):\eta^2\equiv\mathbb{1}\right\}\cup\{(1,\mathbb{1})\}.$$

  We assume that $z_1=0$ and hence also $z_2=0$. In this case η_1 and η_2 are quadratic characters. If $\eta_1=\mathbb{1}$, then

  $$\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})=\mathcal{L}_F^S(\chi,s)^3\,\mathcal{L}_F^S(\mu_2\chi,s)^4.$$

  If $\eta_2=\mathbb{1}$ then $\mathcal{L}^S(s,\pi,\mathbb{1},\mathfrak{st})$ admits a pole of order 7 at $s=1$, while $\mathcal{E}_E(\mathbb{1},f_s,s,g)$ admits a pole at most of order 1 at $s=\frac{1}{2}$ which brings us to a contradiction.

Assuming that $\eta_2 \neq \mathbb{1}$, then $\mathcal{L}^S(s, \pi, \eta_2, \mathfrak{st})$ admits a pole of order 4 at $s = 1$ while $\mathcal{E}_E(\eta_2, f_s, s, g)$ admits a pole at most of order 1 at $s = \frac{1}{2}$ which again brings us to a contradiction.

We now assume that $\eta_1, \eta_2 \not\equiv \mathbb{1}$ are quadratic characters. In this case

$$\mathcal{L}^S(s, \pi, \chi, \mathfrak{st}) = \mathcal{L}_F^S(\eta_1\chi, s)^2 \mathcal{L}_F^S(\eta_2\chi, s)^2 \mathcal{L}_F^S(\eta_1\eta_2\chi, s)^2 \mathcal{L}_F^S(\chi, s).$$

$\mathcal{L}^S(s, \pi, \eta_1, \mathfrak{st})$ admits a pole at least of order 2 at $s = 1$, while $\mathcal{E}_E(\eta_1, f_s, s, g)$ admits a pole at most of order 1, which again brings us to a contradiction.

In conclusion, $z_1 = 1$ and hence also $z_2 \geq \frac{1}{2}$. In particular, $z_2 = 1$. We conclude that $\eta_1 \equiv \eta_2 \equiv \mathbb{1}$ which proves the assertion.

- $E = F \times K$, where K/F is a quadratic extension: Since $\mathcal{E}_E\left(\mu_1 |\cdot|^{-1}, f_s, s, g\right)$ and $\mathcal{E}_E\left(\mu_2 |\cdot|^{-1}, f_s, s, g\right)$ admits a pole at $s = 2$, it holds that

$$(z_1, \eta_1), (z_2, \eta_2) \in \{(0, \mathbb{1}), (0, \chi_K), (1, \mathbb{1}), (1, \chi_K)\}.$$

The proof that $z_1, z_2 \neq 0$ is similar to the split case. It then holds that $z_1 = z_2 = 1$. We need to prove that $\eta_1 \equiv \eta_2 \equiv \mathbb{1}$ or $\eta_1 \equiv \eta_2 \equiv \chi_K$ cannot happen.

Assume that $\eta_1 \equiv \eta_2 \equiv \mathbb{1}$; in this case

$$\mathcal{L}^S(s, \pi, \chi, \mathfrak{st}) = \mathcal{L}_F^S(\chi, s)^3 \mathcal{L}_F^S(\chi, s-1)^2 \mathcal{L}_F^S(\chi, s+1)^2.$$

$\mathcal{L}^S(s, \pi, \mathbb{1}, \mathfrak{st})$ would have a pole of order at least 3 at $s = 1$ while $\mathcal{E}_E(\mathbb{1}, f_s, s, g)$ admits a pole of order at most 1 at $s = \frac{1}{2}$, which brings us to a contradiction.

Assume that $\eta_1 \equiv \eta_2 \equiv \chi_K$; in this case

$$\mathcal{L}^S(s, \pi, \chi, \mathfrak{st}) = \mathcal{L}_F^S(\chi, s)^3 \mathcal{L}_F^S(\chi_K\chi, s-1)^2 \mathcal{L}_F^S(\chi_K\chi, s+1)^2.$$

$\mathcal{L}^S(s, \pi, \mathbb{1}, \mathfrak{st})$ would have a pole at least of order 3 at $s = 1$ while $\mathcal{E}_E(\mathbb{1}, f_s, s, g)$ admits a pole at most of order 1 at $s = \frac{1}{2}$, which brings us to a contradiction.

In conclusion, $\mu_1 = |\cdot|$ and $\mu_2 = |\cdot| \chi_K$, or vice versa, which proves the assertion.

- E/F is a cubic Galois extension: Since $\mathcal{E}_E\left(\mu_1\left|\cdot\right|^{-1}, f_s, s, g\right)$ and $\mathcal{E}_E\left(\mu_2\left|\cdot\right|^{-1}, f_s, s, g\right)$ admit a pole at $s = 2$, it holds that

$$(z_1, \eta_1), (z_2, \eta_2) \in \left\{(0, \eta) : \eta^2 \equiv \mathbb{1}\right\} \cup \left\{(1, \chi_E)\right\}.$$

The proof that $(z_1, \eta_1), (z_2, \eta_2) \neq (0, \eta)$ for η a quadratic character is similar to the split case. Hence, $\mu_1 \equiv \mu_2 \equiv \chi_E \left|\cdot\right|$ which proves the assertion.

□

Appendix A

The Convolution $\left(F^{*}\left(\Psi_{E},\mathbb{1},\cdot,s\right)*P_{s}\right)$

In this appendix we compute the convolution $\left(F^{*}\left(\Psi_{E},\mathbb{1},\cdot,s\right)*P_{s}\right)(g)$ for any g. We retain the notations of Chapter 4. We recall that

$$R_0\left(q^{-s-\frac{1}{2}}\right)=q^{-4s-4}+q^{-3s-\frac{7}{2}}+q^{-3s-\frac{5}{2}}+q^{-2s-2}+q^{-s-\frac{3}{2}}+q^{-s-\frac{1}{2}}+1,\quad R_1\left(q^{-s-\frac{1}{2}}\right)=q^{-2s-2}.$$

We recall that

$$\begin{aligned}&\left(F^{*}\left(\Psi_{E},\mathbb{1},\cdot,s\right)*P_{s}\right)(g)\\&=N_E(s)\left(\left(R_1-q^{-3}Q\right)F\left(\Psi_E,\mathbb{1},g,s\right)-q^{-3}Q\left(F\left(\Psi_E,\mathbb{1},\cdot,s\right)*\mathbb{1}_{K\omega_2^{\vee}(\varpi)K}\right)(g)\right),\end{aligned}$$

where

$$N_E(s)=\begin{cases}\frac{\xi_F\left(s+\frac{5}{2}\right)\xi_E\left(s+\frac{3}{2}\right)\xi_F(2s+1)}{\xi_F\left(s+\frac{3}{2}\right)^2\xi_F\left(s+\frac{7}{2}\right)\xi_F\left(s+\frac{1}{2}\right)}, & E\text{ is a field}\\ \frac{\xi_F\left(s+\frac{5}{2}\right)\xi_K\left(s+\frac{3}{2}\right)\xi_F(2s+1)}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)\xi_F\left(s+\frac{1}{2}\right)}, & E=F\times K,\ K\text{ is a field}\end{cases}$$

A.1 E is a Field

A.1.1 Toric Elements

For $t_1,t_2\in F^{\times}$ we denote $|t_1|=q^{-n}$ and $|t_2|=q^{-m}$. Let $g=h_{\alpha}(t_1)\,h_{\beta}(t_2)$. We compute $\left(F^{*}\left(\Psi_{E},\mathbb{1},\cdot,s\right)*P_{s}(\cdot)\right)(g)$.

1. $\left|\frac{t_2}{t_1^2}\right| \leq \frac{1}{q}$: In this case

$$F(\Psi_E, \mathbb{1}, g, s) = \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m(2s+4)-n\left(-3s-\frac{9}{2}\right)} \left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right).$$

   (a) $\left|\frac{t_1^3}{t_2}\right| \leq \frac{1}{q^2}$: In this case

$$GS(\Psi_E, g) = q^3 - 1$$

   and also

$$\begin{aligned}
&\left(F(\Psi_E, \mathbb{1}, \cdot, s) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m(2s+4)-n\left(-3s-\frac{9}{2}\right)} \left(q^{s+\frac{7}{2}}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\right. \\
&+ q^{\frac{5}{2}-s}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{\frac{1}{2}-s}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right) \\
&+ q^{-2s}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)}\right) + q^{2s+5}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right) \\
&+ (q-1)q^{\frac{1}{2}-s}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right) + q^{s+\frac{9}{2}}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right) \\
&\left.+ (q-1)q^{-2s}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\right) + GS(\Psi_E, g)\, F(\Psi_E, \mathbb{1}, g, s).
\end{aligned}$$

   Hence

$$\left(F^*(\Psi_E, \mathbb{1}, \cdot, s) * P_s\right)(g) = 0.$$

   (b) $\left|\frac{t_1^3}{t_2}\right| = \frac{1}{q}$: In this case

$$GS(\Psi_E, g) = q^3 - 1$$

   and also

$$\begin{aligned}
&\left(F(\Psi_E, \mathbb{1}, \cdot, s) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m(2s+4)-n\left(-3s-\frac{9}{2}\right)} \left(q^{s+\frac{7}{2}}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\right. \\
&+ q^{\frac{5}{2}-s}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{-2s}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)}\right) \\
&+ q^{2s+5}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{s+\frac{9}{2}}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right) \\
&\left.+ (q-1)q^{-2s}\left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\right) + GS(\Psi_E, g)\, F(\Psi_E, \mathbb{1}, g, s).
\end{aligned}$$

   Hence

$$\left(F^*(\Psi_E, \mathbb{1}, \cdot, s) * P_s\right)(g) = 0.$$

(c) $\left|\frac{t_1^3}{t_2}\right| = 1$: In this case

$$GS\left(\Psi_E, g\right) = -1$$

and also

$$\begin{aligned}&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)\\ &= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(q^{\frac{5}{2}-s}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right)\right.\\ &\left.+q^{2s+5}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right)+q^{s+\frac{9}{2}}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)\right)\\ &+GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).\end{aligned}$$

Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = 0.$$

2. $\left|\frac{t_2}{t_1^2}\right| = 1$: In this case

$$F\left(\Psi_E, \mathbb{1}, g, s\right) = \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right).$$

(a) $\left|\frac{t_1^3}{t_2}\right| \leq \frac{1}{q^2}$: In this case

$$GS\left(\Psi_E, g\right) = q^3 - 1$$

and also

$$\begin{aligned}&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)\\ &= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}\left(q^{s+\frac{7}{2}-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\right.\\ &+q^{\frac{5}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right)\\ &+q^{\frac{1}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right)\\ &+q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\\ &+q^{\frac{1}{2}-s-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right)\\ &+(q-1)q^{\frac{1}{2}-s-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right)\\ &+q^{-2s-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\end{aligned}$$

$$+ (q-1)q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\Big)$$
$$+ GS\left(\Psi_E, g\right)F\left(\Psi_E, \mathbb{1}, g, s\right).$$

Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = \frac{q^{-\left(s+\frac{7}{2}\right)n}}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)}.$$

(b) $\left|\frac{t_1^3}{t_2}\right| = \frac{1}{q}$: In this case

$$GS\left(\Psi_E, g\right) = q^3 - 1$$

and also

$$\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)$$
$$= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}\Big(q^{s+\frac{7}{2}-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)$$
$$+ q^{\frac{5}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right)$$
$$+ q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)$$
$$+ q^{-2s-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)$$
$$+ (q-1)q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\Big)$$
$$+ GS\left(\Psi_E, g\right)F\left(\Psi_E, \mathbb{1}, g, s\right).$$

Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = \frac{q^{-\left(s+\frac{7}{2}\right)}}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)}.$$

(c) $\left|\frac{t_1^3}{t_2}\right| = 1$: In this case

$$GS\left(\Psi_E, g\right) = -q^2 - 1$$

and also

$$\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)$$
$$= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}\left(q^{\frac{5}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right)\right)$$
$$+ GS\left(\Psi_E, g\right)F\left(\Psi_E, \mathbb{1}, g, s\right).$$

Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = \frac{1}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)}.$$

3. $\left|\frac{t_2}{t_1^2}\right| = q$: In this case

$$F\left(\Psi_E, \mathbb{1}, g, s\right) = \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right).$$

   (a) $\left|\frac{t_2^2}{t_1^3}\right| \leq \frac{1}{q^2}$: In this case

$$GS\left(\Psi_E, g\right) = q^3 - 1$$

   and hence

$$\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}\left(q^{s+\frac{7}{2}-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\right. \\
&+ q^{\frac{5}{2}-s-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right) \\
&+ q^{\frac{1}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right) \\
&+ q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right) \\
&+ q^{\frac{1}{2}-s-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right) \\
&+ (q-1)q^{\frac{1}{2}-s-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right) \\
&+ q^{-2s-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right) \\
&\left.+ (q-1)q^{-2s-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\right) \\
&+ GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).
\end{aligned}$$

   Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = 0.$$

   (b) $\left|\frac{t_2^2}{t_1^3}\right| = \frac{1}{q}$: In this case

$$GS\left(\Psi_E, g\right) = q^3 - 1$$

and also

$$\begin{aligned}
&\left(F(\Psi_E, \mathbb{1}, \cdot, s) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)\\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}\left(q^{s+\frac{7}{2}-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\right.\\
&+ q^{\frac{5}{2}-s-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right)\\
&+ q^{\frac{1}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right)\\
&+ q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\\
&+ q^{-2s-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\\
&\left.+(q-1)q^{-2s-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\right)\\
&+ GS(\Psi_E, g)\, F(\Psi_E, \mathbb{1}, g, s).
\end{aligned}$$

Hence

$$(F^*(\Psi_E, \mathbb{1}, \cdot, s) * P_s)(g) = 0.$$

(c) $\left|\frac{t_2^2}{t_1^3}\right| = 1$: In this case

$$GS(\Psi_E, g) = -1$$

and also

$$\begin{aligned}
&\left(F(\Psi_E, \mathbb{1}, \cdot, s) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)\\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}\left(q^{\frac{5}{2}-s-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right)\right.\\
&+ q^{\frac{1}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right)\\
&\left.+ q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\right)\\
&+ GS(\Psi_E, g)\, F(\Psi_E, \mathbb{1}, g, s).
\end{aligned}$$

Hence

$$(F^*(\Psi_E, \mathbb{1}, \cdot, s) * P_s)(g) = 0.$$

4. $\left|\frac{t_2}{t_1^2}\right| > q$: In this case

$$F(\Psi_E, \mathbb{1}, g, s) = \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right).$$

(a) $\left|\frac{t_2^2}{t_1^3}\right| \le \frac{1}{q^2}$: In this case

$$GS\left(\Psi_E, g\right) = q^3 - 1$$

and also

$$\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)} \left(q^{s+\frac{7}{2}}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\right. \\
&+ q^{\frac{5}{2}-s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{2s+5}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right) \\
&+ q^{s+\frac{9}{2}}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right) + q^{\frac{1}{2}-s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right) \\
&+ (q-1)q^{\frac{1}{2}-s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right) + q^{-2s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right) \\
&\left.+(q-1)q^{-2s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\right) + GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).
\end{aligned}$$

Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = 0.$$

(b) $\left|\frac{t_2^2}{t_1^3}\right| = \frac{1}{q}$: In this case

$$GS\left(\Psi_E, g\right) = q^3 - 1$$

and also

$$\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)} \left(q^{s+\frac{7}{2}}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\right. \\
&+ q^{\frac{5}{2}-s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{2s+5}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right) \\
&+ q^{s+\frac{9}{2}}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right) + q^{-2s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right) \\
&\left.+(q-1)q^{-2s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\right) + GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).
\end{aligned}$$

Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = 0.$$

(c) $\left|\frac{t_2^2}{t_1^3}\right| = 1$: In this case

$$GS\left(\Psi_E, g\right) = -1$$

and also

$$\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^{\vee}(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)} \left(q^{\frac{5}{2}-s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right)\right. \\
&\left. + q^{2s+5}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{s+\frac{9}{2}}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)\right) \\
&+ GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).
\end{aligned}$$

Hence

$$\left(F^{*}\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = 0.$$

A.1.2 Non-Toric Elements

For $t_1, t_2 \in F^{\times}$ and $d \in F$ we denote $|t_1| = q^{-n}$, $|t_2| = q^{-m}$ and $|d| = q^{l}$. Let $g = x_{\alpha}(d)\, h_{\alpha}(t_1)\, h_{\beta}(t_2)$. We compute $\left(F^{*}\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s(\cdot)\right)(g)$.

1. $\underline{\left|\frac{dt_1^2}{t_2}\right| \le 1}$: In this case

$$F\left(\Psi_E, \mathbb{1}, g, s\right) = \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)} \left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right)$$

   (a) $\underline{\left|\frac{t_2^2}{t_1^3}\right| \le \frac{1}{q^2}}$: In this case

$$GS\left(\Psi_E, g\right) = q^3 - 1$$

   and also

$$\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^{\vee}(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)} \left(q^{s+\frac{7}{2}}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\right. \\
&+ q^{\frac{5}{2}-s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{2s+5}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right) \\
&+ q^{s+\frac{9}{2}}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right) + q^{\frac{3}{2}-s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right) \\
&\left.+q^{1-2s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\right) + GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).
\end{aligned}$$

   Hence

$$\left(F^{*}\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = 0.$$

(b) $\left|\frac{t_2^2}{t_1^3}\right| = \frac{1}{q}$: In this case

$$GS\left(\Psi_E, g\right) = q^3 - 1$$

and also

$$\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)} \left(q^{s+\frac{7}{2}}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\right. \\
&+ q^{\frac{5}{2}-s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{2s+5}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right) \\
&\left.+ q^{s+\frac{9}{2}}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right) + q^{1-2s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)}\right)\right) \\
&+ GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).
\end{aligned}$$

Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = 0.$$

(c) $\left|\frac{t_2^2}{t_1^3}\right| = \frac{1}{q}$: In this case

$$GS\left(\Psi_E, g\right) = -1$$

and also

$$\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m\left(-s-\frac{1}{2}\right)-n\left(3s+\frac{9}{2}\right)} \left(q^{\frac{5}{2}-s}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right)\right. \\
&\left.+ q^{2s+5}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{s+\frac{9}{2}}\left(1-q^{(3n-2m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)\right) \\
&+ GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).
\end{aligned}$$

Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = 0.$$

2. $\left|\frac{dt_1^2}{t_2}\right| > 1$: In this case

$$F\left(\Psi_E, \mathbb{1}, g, s\right) = \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m(2s+4)+(l-n)\left(-3s-\frac{9}{2}\right)} \left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right)$$

(a) $\left|\frac{d^3t_1^3}{t_2}\right|\le\frac{1}{q^2}$: In this case

$$GS(\Psi_E,g)=q^3-1$$

and also

$$\begin{aligned}&\left(F(\Psi_E,\mathbb{1},\cdot,s)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)\\&=\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}q^{-m(2s+4)+(l-n)\left(-3s-\frac{9}{2}\right)}\left(q^{s+\frac{7}{2}}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)}\right)\right.\\&+q^{\frac{5}{2}-s}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-2s-3}\right)+q^{2s+5}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-2s-3}\right)\\&+q^{s+\frac{9}{2}}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)+q^{\frac{3}{2}-s}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right)\\&\left.+q^{1-2s}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)}\right)\right)+GS(\Psi_E,g)\,F(\Psi_E,\mathbb{1},g,s).\end{aligned}$$

Hence

$$\left(F^*(\Psi_E,\mathbb{1},\cdot,s)*P_s\right)(g)=0.$$

(b) $\left|\frac{t_2^2}{t_1^3}\right|=\frac{1}{q}$: In this case

$$GS(\Psi_E,g)=q^3-1$$

and also

$$\begin{aligned}&\left(F(\Psi_E,\mathbb{1},\cdot,s)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)\\&=\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}q^{-m(2s+4)+(l-n)\left(-3s-\frac{9}{2}\right)}\left(q^{s+\frac{7}{2}}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)}\right)\right.\\&+q^{\frac{5}{2}-s}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-2s-3}\right)+q^{2s+5}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-2s-3}\right)\\&\left.+q^{s+\frac{9}{2}}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)+q^{1-2s}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)}\right)\right)\\&+GS(\Psi_E,g)\,F(\Psi_E,\mathbb{1},g,s).\end{aligned}$$

Hence

$$\left(F^*(\Psi_E,\mathbb{1},\cdot,s)*P_s\right)(g)=0.$$

(c) $\left|\frac{t_2^2}{t_1^3}\right|=1$: In this case

$$GS(\Psi_E,g)=-1$$

and also

$$\left(F\left(\Psi_E,\mathbb{1},\cdot,s\right)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)$$
$$=\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}q^{-m(2s+4)+(l-n)\left(-3s-\frac{9}{2}\right)}\left(q^{\frac{5}{2}-s}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-2s-3}\right)\right.$$
$$\left.+q^{2s+5}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-2s-3}\right)+q^{s+\frac{9}{2}}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)\right)$$
$$+GS\left(\Psi_E,g\right)F\left(\Psi_E,\mathbb{1},g,s\right).$$

Hence

$$\left(F^*\left(\Psi_E,\mathbb{1},\cdot,s\right)*P_s\right)(g)=0.$$

A.2 $E = F \times K$, K is a Field

A.2.1 Toric Elements

For $t_1, t_2 \in F^\times$ we denote $|t_1| = q^{-n}$ and $|t_2| = q^{-m}$. Let $g = h_\alpha(t_1)\,h_\beta(t_2)$. We compute $\left(F^*\left(\Psi_E,\mathbb{1},\cdot,s\right)*P_s(\cdot)\right)(g)$.

1. $\left|\frac{t_2}{t_1^2}\right| \le \frac{1}{q}$: In this case

$$F\left(\Psi_E,\mathbb{1},g,s\right)=\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}q^{-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right)$$

(a) $\left|\frac{t_1^3}{t_2}\right| \le \frac{1}{q^2}$: In this case

$$GS\left(\Psi_E,g\right)=q^3-1$$

and also

$$\left(F\left(\Psi_E,\mathbb{1},\cdot,s\right)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)$$
$$=\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}q^{-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(q^{s+\frac{7}{2}}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\right.$$
$$+q^{\frac{5}{2}-s}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right)+q^{\frac{1}{2}-s}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-s+\frac{7}{2}}\right)$$
$$+q^{2s+5}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right)+(q-1)q^{\frac{1}{2}-s}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-s+\frac{7}{2}}\right)$$

$$+q^{-2s}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)+q^{s+\frac{9}{2}}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)$$
$$+(q-1)q^{-2s}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\Big)+GS(\Psi_E,g)\,F(\Psi_E,\mathbb{1},g,s).$$

Hence

$$(F^*(\Psi_E,\mathbb{1},\cdot,s)*P_s)(g)=0.$$

(b) $\left|\frac{t_1^3}{t_2}\right|=\frac{1}{q}$: In this case

$$GS(\Psi_E,g)=q^3-1$$

and also

$$\left(F(\Psi_E,\mathbb{1},\cdot,s)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)$$
$$=\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}q^{-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\Big(q^{s+\frac{7}{2}}\left(1-q^{((m-3n)\left(s+\frac{3}{2}\right)}\right)$$
$$+q^{\frac{5}{2}-s}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right)+q^{2s+5}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right)$$
$$+q^{-2s}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)+q^{s+\frac{9}{2}}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)$$
$$+(q-1)q^{-2s}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\Big)+GS(\Psi_E,g)\,F(\Psi_E,\mathbb{1},g,s).$$

Hence

$$(F^*(\Psi_E,\mathbb{1},\cdot,s)*P_s)(g)=0.$$

(c) $\left|\frac{t_1^3}{t_2}\right|=1$: In this case

$$GS(\Psi_E,g)=-1$$

and also

$$\left(F(\Psi_E,\mathbb{1},\cdot,s)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)$$
$$=\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}q^{-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\Big(q^{\frac{5}{2}-s}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right)$$
$$+q^{2s+5}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right)+q^{s+\frac{9}{2}}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)\Big)$$
$$+GS(\Psi_E,g)\,F(\Psi_E,\mathbb{1},g,s).$$

Hence

$$(F^*(\Psi_E,\mathbb{1},\cdot,s)*P_s)(g)=0.$$

2. $\left|\frac{t_2}{t_1^2}\right| = 1$: In this case

$$F(\Psi_E, \mathbb{1}, g, s) = \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m(2s+4)-n\left(-3s-\frac{9}{2}\right)} \left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right)$$

(a) $\left|\frac{t_1^3}{t_2}\right| \leq \frac{1}{q^2}$: In this case

$$GS(\Psi_E, g) = q^3 - 1$$

and also

$$\begin{aligned}
&\left(F(\Psi_E, \mathbb{1}, \cdot, s) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} \left(q^{s+\frac{7}{2}-m(2s+4)-n\left(-3s-\frac{9}{2}\right)} \left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\right. \\
&+ q^{\frac{5}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)} \left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right) \\
&+ q^{\frac{1}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)} \left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)-s+\frac{7}{2}}\right) \\
&+ q^{2-m-n\left(s+\frac{3}{2}\right)} \left(1 - q^{(n-m)\left(s+\frac{3}{2}\right)}\right) + \\
&\qquad (q-1) q^{\frac{1}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)} \left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)-s+\frac{7}{2}}\right) \\
&+ q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)} \left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)}\right) \\
&+ q^{\frac{3}{2}-s-m-n\left(s+\frac{3}{2}\right)} \left(1 - q^{(n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right) \\
&\left.+ (q-1) q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)} \left(1 - q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\right) \\
&+ GS(\Psi_E, g) F(\Psi_E, \mathbb{1}, g, s).
\end{aligned}$$

Hence

$$\left(F^*(\Psi_E, \mathbb{1}, \cdot, s) * P_s\right)(g) = \frac{q^{-\left(s+\frac{7}{2}\right)n}}{\xi_F\left(s+\frac{7}{2}\right)}.$$

(b) $\left|\frac{t_1^3}{t_2}\right| = \frac{1}{q}$: In this case

$$GS(\Psi_E, g) = q^3 - 1$$

and also

$$\left(F(\Psi_E, \mathbb{1}, \cdot, s) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)$$

$$= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}\left(q^{s+\frac{7}{2}-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\right.$$
$$+q^{\frac{5}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)-2s-3}\right)$$
$$+q^{2-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)}\right)$$
$$+q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)$$
$$+q^{\frac{3}{2}-s-(3n-1)-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right)$$
$$\left.+(q-1)q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)\right)$$
$$+GS\left(\Psi_E,g\right)F\left(\Psi_E,\mathbb{1},g,s\right).$$

Hence

$$\left(F^*\left(\Psi_E,\mathbb{1},\cdot,s\right)*P_s\right)(g)=\frac{q^{-\left(s+\frac{7}{2}\right)}}{\xi_F\left(s+\frac{7}{2}\right)}.$$

(c) $\left|\frac{t_1^3}{t_2}\right|=1$: In this case

$$GS\left(\Psi_E,g\right)=-1$$

and also

$$\left(F\left(\Psi_E,\mathbb{1},\cdot,s\right)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)$$
$$=\frac{\xi\left(s+\frac{3}{2}\right)q^{\frac{5}{2}-s}}{\xi\left(s+\frac{5}{2}\right)\xi\left(2s+3\right)}+\frac{q^{\frac{3}{2}-s}}{\xi\left(s+\frac{5}{2}\right)}+GS\left(\Psi_E,g\right)F\left(\Psi_E,\mathbb{1},g,s\right).$$

Hence

$$\left(F^*\left(\Psi_E,\mathbb{1},\cdot,s\right)*P_s\right)(g)=\frac{1}{\xi_F\left(s+\frac{7}{2}\right)}.$$

3. $\left|\frac{t_2}{t_1^2}\right|=q$: In this case

$$F\left(\Psi_E,\mathbb{1},g,s\right)=\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}q^{-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right)$$

(a) $\left|\frac{t_2}{t_1}\right|\leq\frac{1}{q^2}$: In this case

$$GS\left(\Psi_E,g\right)=q^3-1$$

and also

$$
\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}\left(q^{s+\frac{7}{2}-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)}\right)\right. \\
&+ q^{\frac{5}{2}-s-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-2s-3}\right) \\
&+ q^{\frac{1}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right) \\
&+ q^{2-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)}\right) \\
&+ (q-1)q^{\frac{1}{2}-s-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right) \\
&+ q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right) \\
&+ q^{\frac{3}{2}-s-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right) \\
&\left.+ (q-1)q^{-2s-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)}\right)\right) \\
&+ GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).
\end{aligned}
$$

Hence

$$
\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = \frac{q^{1-n\left(s+\frac{7}{2}\right)}}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)}.
$$

(b) $\underline{\left|\frac{t_2}{t_1}\right| = \frac{1}{q}}$: In this case

$$
GS\left(\Psi_E, g\right) = q^3 - 1
$$

and also

$$
\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}\left(q^{s+\frac{7}{2}-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)}\right)\right. \\
&+ q^{\frac{5}{2}-s-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-2s-3}\right) \\
&+ q^{\frac{1}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right) \\
&+ q^{2-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)}\right) \\
&+ q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)
\end{aligned}
$$

$$+ q^{\frac{3}{2}-s-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right)$$
$$+ (q-1)q^{-2s-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)}\right)\Big)$$
$$+ GS(\Psi_E, g) F(\Psi_E, \mathbb{1}, g, s).$$

Hence

$$(F^*(\Psi_E, \mathbb{1}, \cdot, s) * P_s)(g) = \frac{q^{1-2\left(s+\frac{7}{2}\right)}}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)}.$$

(c) $\left|\frac{t_2}{t_1}\right| = 1$: In this case

$$GS(\Psi_E, g) = q^2 - 1$$

and also

$$\left(F(\Psi_E, \mathbb{1}, \cdot, s) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)$$
$$= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}\left(q^{\frac{5}{2}-s-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-2s-3}\right)\right.$$
$$+ q^{\frac{1}{2}-s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right)$$
$$+ q^{-2s-m(2s+4)-n\left(-3s-\frac{9}{2}\right)}\left(1-q^{(m-3n)\left(s+\frac{3}{2}\right)}\right)$$
$$\left.+ q^{\frac{3}{2}-s-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right)\right)$$
$$+ GS(\Psi_E, g) F(\Psi_E, \mathbb{1}, g, s).$$

Hence

$$(F^*(\Psi_E, \mathbb{1}, \cdot, s) * P_s)(g) = \frac{q^{1-\left(s+\frac{7}{2}\right)}}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)}.$$

4. $\left|\frac{t_2}{t_1^2}\right| > q$: In this case

$$F(\Psi_E, \mathbb{1}, g, s) = \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m-n\left(s+\frac{3}{2}\right)}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right)$$

(a) $\left|\frac{t_2}{t_1}\right| \leq \frac{1}{q^2}$: In this case

$$GS(\Psi_E, g) = q^3 - 1$$

and also

$$\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m-n\left(s+\frac{3}{2}\right)} \left(q^{s+\frac{7}{2}}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)}\right)\right. \\
&+ q^{\frac{5}{2}-s}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{s+\frac{7}{2}}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right) \\
&+ q^2\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)}\right) + (q-1)q^{\frac{1}{2}-s}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right) \\
&+ q^3\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{\frac{3}{2}-s}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right) \\
&\left.+ (q-1)q^{-2s}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)}\right)\right) + GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).
\end{aligned}$$

Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = \frac{q^{2n-m-\left(s+\frac{7}{2}\right)n}}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)}.$$

(b) $\left|\frac{t_2}{t_1}\right| = \frac{1}{q}$: In this case

$$GS\left(\Psi_E, g\right) = q^3 - 1$$

and also

$$\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m-n\left(s+\frac{3}{2}\right)} \left(q^{s+\frac{7}{2}}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)}\right)\right. \\
&+ q^{\frac{5}{2}-s}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{s+\frac{7}{2}}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right) \\
&+ q^2\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)}\right) + q^3\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-2s-3}\right) \\
&\left.+ q^{\frac{3}{2}-s}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right) + (q-1)q^{-2s}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)}\right)\right) \\
&+ GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).
\end{aligned}$$

Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = \frac{q^{n-1-\left(s+\frac{7}{2}\right)n}}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)}.$$

(c) $\left|\frac{t_2}{t_1}\right| = 1$: In this case

$$GS\left(\Psi_E, g\right) = q^2 - 1$$

and also

$$\begin{aligned}&\left(F(\Psi_E,\mathbb{1},\cdot,s)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)\\&=\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}q^{-m-n\left(s+\frac{3}{2}\right)}\left(q^{\frac{5}{2}-s}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-2s-3}\right)\right.\\&+q^{s+\frac{7}{2}}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right)+q^{3}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-2s-3}\right)\\&\left.+q^{\frac{3}{2}-s}\left(1-q^{(n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right)\right)+GS(\Psi_E,g)\,F(\Psi_E,\mathbb{1},g,s).\end{aligned}$$

Hence

$$(F^{*}(\Psi_E,\mathbb{1},\cdot,s)*P_s)(g)=\frac{q^{n-\left(s+\frac{7}{2}\right)n}}{\xi_F\left(s+\frac{3}{2}\right)\xi_F\left(s+\frac{7}{2}\right)}.$$

A.2.2 Non-Toric Elements

For $t_1,t_2\in F^{\times}$ and $d\in F$ we denote $|t_1|=q^{-n}$, $|t_2|=q^{-m}$ and $|d|=q^{l}$. Let $g=x_\alpha(d)\,h_\alpha(t_1)\,h_\beta(t_2)$. We compute $(F^{*}(\Psi_E,\mathbb{1},\cdot,s)*P_s(\cdot))(g)$.

1. $\left|\frac{dt_1^2}{t_2}\right|\le 1$: In this case

$$F(\Psi_E,\mathbb{1},g,s)=\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}q^{-m-(l+n)\left(s+\frac{3}{2}\right)}\left(1-q^{(l+n-m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right).$$

   (a) $\left|\frac{dt_2}{t_1}\right|\le\frac{1}{q^2}$: In this case

$$GS(\Psi_E,g)=q^3-1$$

   and also

$$\begin{aligned}&\left(F(\Psi_E,\mathbb{1},\cdot,s)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)\\&=\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}q^{-m-(l+n)\left(s+\frac{3}{2}\right)}\left(q^{s+\frac{7}{2}}\left(1-q^{(l+n-m)\left(s+\frac{3}{2}\right)}\right)\right.\\&+q^{\frac{5}{2}-s}\left(1-q^{(l+n-m)\left(s+\frac{3}{2}\right)-2s-3}\right)+q^{2s+5}\left(1-q^{(l+n-m)\left(s+\frac{3}{2}\right)-2s-3}\right)\\&+q^{s+\frac{9}{2}}\left(1-q^{(l+n-m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)+q^{\frac{3}{2}-s}\left(1-q^{(l+n-m)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right)\\&\left.+q^{1-2s}\left(1-q^{(l+n-m)\left(s+\frac{3}{2}\right)}\right)\right)+GS(\Psi_E,g)\,F(\Psi_E,\mathbb{1},g,s).\end{aligned}$$

   Hence

$$(F^{*}(\Psi_E,\mathbb{1},\cdot,s)*P_s)(g)=0.$$

(b) $\left|\frac{dt_2}{t_1}\right| = \frac{1}{q}$: In this case

$$GS\left(\Psi_E, g\right) = q^3 - 1$$

and also

$$\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m-(l+n)\left(s+\frac{3}{2}\right)} \left(q^{s+\frac{7}{2}}\left(1 - q^{(l+n-m)\left(s+\frac{3}{2}\right)}\right)\right. \\
&+ q^{\frac{5}{2}-s}\left(1 - q^{(l+n-m)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{2s+5}\left(1 - q^{(l+n-m)\left(s+\frac{3}{2}\right)-2s-3}\right) \\
&\left.+ q^{s+\frac{9}{2}}\left(1 - q^{(l+n-m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right) + q^{1-2s}\left(1 - q^{(l+n-m)\left(s+\frac{3}{2}\right)}\right)\right) \\
&+ GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).
\end{aligned}$$

Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = 0.$$

(c) $\left|\frac{dt_2}{t_1}\right| = 1$: In this case

$$GS\left(\Psi_E, g\right) = -1$$

and also

$$\begin{aligned}
&\left(F\left(\Psi_E, \mathbb{1}, \cdot, s\right) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) \\
&= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m-(l+n)\left(s+\frac{3}{2}\right)} \left(q^{\frac{5}{2}-s}\left(1 - q^{(l+n-m)\left(s+\frac{3}{2}\right)-2s-3}\right)\right. \\
&\left.+ q^{2s+5}\left(1 - q^{(l+n-m)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{s+\frac{9}{2}}\left(1 - q^{(l+n-m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)\right) \\
&+ GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).
\end{aligned}$$

Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = 0.$$

2. $\left|\frac{dt_1^2}{t_2}\right| > 1$: In this case

$$F\left(\Psi_E, \mathbb{1}, g, s\right) = \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m(2s+4)+(l-n)\left(-3s-\frac{9}{2}\right)}\left(1 - q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-s-\frac{3}{2}}\right).$$

(a) $\left|\frac{dt_1^3}{t_2}\right|\leq\frac{1}{q^2}$: In this case

$$GS\left(\Psi_E,g\right)=q^3-1$$

and also

$$\begin{aligned}&\left(F\left(\Psi_E,\mathbb{1},\cdot,s\right)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)\\&=\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}q^{-m(2s+4)+(l-n)\left(-3s-\frac{9}{2}\right)}\left(q^{s+\frac{7}{2}}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)}\right)\right.\\&+q^{\frac{5}{2}-s}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-2s-3}\right)+q^{2s+5}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-2s-3}\right)\\&+q^{s+\frac{9}{2}}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)+q^{\frac{3}{2}-s}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)+s+\frac{3}{2}}\right)\\&\left.+q^{1-2s}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)}\right)\right)+GS\left(\Psi_E,g\right)F\left(\Psi_E,\mathbb{1},g,s\right).\end{aligned}$$

Hence

$$\left(F^*\left(\Psi_E,\mathbb{1},\cdot,s\right)*P_s\right)(g)=0.$$

(b) $\left|\frac{dt_1^3}{t_2}\right|=\frac{1}{q}$: In this case

$$GS\left(\Psi_E,g\right)=q^3-1$$

and also

$$\begin{aligned}&\left(F\left(\Psi_E,\mathbb{1},\cdot,s\right)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)\\&=\frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)}q^{-m(2s+4)+(l-n)\left(-3s-\frac{9}{2}\right)}\left(q^{s+\frac{7}{2}}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)}\right)\right.\\&+q^{\frac{5}{2}-s}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-2s-3}\right)+q^{2s+5}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-2s-3}\right)\\&\left.+q^{s+\frac{9}{2}}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)+q^{1-2s}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)}\right)\right)\\&+GS\left(\Psi_E,g\right)F\left(\Psi_E,\mathbb{1},g,s\right).\end{aligned}$$

Hence

$$\left(F^*\left(\Psi_E,\mathbb{1},\cdot,s\right)*P_s\right)(g)=0.$$

(c) $\left|\frac{dt_1^3}{t_2}\right|=1$: In this case

$$GS\left(\Psi_E,g\right)=-q^2-1$$

and also

$$\left(F\left(\Psi_E,\mathbb{1},\cdot,s\right)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g)$$

$$= \frac{\xi\left(s+\frac{3}{2}\right)}{\xi\left(s+\frac{5}{2}\right)} q^{-m(2s+4)+(l-n)\left(-3s-\frac{9}{2}\right)} \left(q^{\frac{5}{2}-s}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-2s-3}\right)\right.$$
$$+ q^{2s+5}\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-2s-3}\right) + q^{s+\frac{9}{2}}\left.\left(1-q^{(3l-3n+m)\left(s+\frac{3}{2}\right)-3s-\frac{9}{2}}\right)\right)$$
$$+ GS\left(\Psi_E, g\right) F\left(\Psi_E, \mathbb{1}, g, s\right).$$

Hence

$$\left(F^*\left(\Psi_E, \mathbb{1}, \cdot, s\right) * P_s\right)(g) = 0.$$

Appendix B

Tables of Intertwining Operators

In this section we list useful tables containing information about the local intertwining operators, poles of global Gindikin-Karpelevich factors and the exponents of $w^{-1}\chi_s(t)$ in the various cases.

B.0.1 Cubic Extension Case

Assume E is a Cubic Field Extension of F. In this case we denote

$$t = h_{\alpha_1\alpha_3^\sigma\alpha_4^{\sigma^2}}(t_1)\, h_{\alpha_2}(t_2) = h_{\alpha_1}(t_1)\, h_{\alpha_2}(t_2)\, h_{\alpha_3}(t_1^\sigma)\, h_{\alpha_4}\left(t_1^{\sigma^2}\right),$$

where $t_1 \in E^\times, t_2 \in F^\times$.

In the following table we list $w^{-1} \cdot \chi_s(t)$ for the various $w \in W(P_E, H_E)$.

$w \in W(P_E, H_E)$	$w^{-1} \cdot \chi_s(t)$
$[]$	$\chi(t_2) \frac{\lvert t_2\rvert_F^{s+\frac{3}{2}}}{\lvert t_1\rvert_E}$
$[2]$	$\chi\left(\frac{\mathrm{Nm}_{E/F}(t_1)}{t_2}\right) \frac{\lvert t_1\rvert_E^{s+\frac{1}{2}}}{\lvert t_2\rvert_F^{s+\frac{3}{2}}}$
$[2,1]$	$\chi\left(\frac{t_2^2}{\mathrm{Nm}_{E/F}(t_1)}\right) \frac{\lvert t_2\rvert_F^{2s}}{\lvert t_1\rvert_E^{s+\frac{1}{2}}}$
$[2,1,2]$	$\chi\left(\frac{\mathrm{Nm}_{E/F}(t_1)}{t_2^2}\right) \frac{\lvert t_1\rvert_E^{s-\frac{1}{2}}}{\lvert t_2\rvert_F^{2s}}$
$[2,1,2,1]$	$\chi\left(\frac{t_2}{\mathrm{Nm}_{E/F}(t_1)}\right) \frac{\lvert t_2\rvert_F^{s-\frac{3}{2}}}{\lvert t_1\rvert_E^{s-\frac{1}{2}}}$
$[2,1,2,1,2]$	$\chi\left(\frac{1}{t_2}\right) \frac{1}{\lvert t_1\rvert_E \lvert t_2\rvert_F^{s-\frac{3}{2}}}$

Table B.1: $w^{-1} \cdot \chi_s$ for $w \in W(P_E, H_E)$, E is a field

In the following table we list the Gindikin-Karpelevich factor $J(w, \chi, \lambda)$ and the poles of the global Gindikin-Karpelevich factor $J(w, \chi_s)$ for $\mathfrak{Re}(s) > 0$.

		$s=\frac{1}{2}$			$s=\frac{3}{2}$		$s=\frac{5}{2}$
$w \in W(P_E, H_E)$	$J(w, \chi_s)$	$\mathbb{1}$	χ_E	$\chi^2=\mathbb{1}$	$\mathbb{1}$	χ_E	$\mathbb{1}$
$[]$	1	0	0	0	0	0	0
$[2]$	$\frac{\mathcal{L}_F\left(s+\frac{3}{2},\chi\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)}$	0	0	0	0	0	0

		$s=\frac{1}{2}$			$s=\frac{3}{2}$		$s=\frac{5}{2}$
$w \in W(P_E, H_E)$	$J(w, \chi_s)$	$\mathbb{1}$	χ_E	$\chi^2=\mathbb{1}$	$\mathbb{1}$	χ_E	$\mathbb{1}$
$[2,1]$	$\frac{\mathcal{L}_F\left(s+\frac{3}{2},\chi\right)\mathcal{L}_E\left(s+\frac{1}{2},\chi\circ\mathrm{Nm}\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_E\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)}$	1	1	0	0	0	0
$[2,1,2]$	$\frac{\mathcal{L}_F\left(s+\frac{3}{2},\chi\right)\mathcal{L}_E\left(s+\frac{1}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s,\chi^2\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_E\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s+1,\chi^2\right)}$	2	1	1	0	0	0
$[2,1,2,1]$	$\frac{\mathcal{L}_F\left(s+\frac{3}{2},\chi\right)\mathcal{L}_E\left(s-\frac{1}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s,\chi^2\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_E\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s+1,\chi^2\right)}$	2	1	1	1	1	0
$[2,1,2,1,2]$	$\frac{\mathcal{L}_F\left(s-\frac{3}{2},\chi\right)\mathcal{L}_F\left(s+\frac{3}{2},\chi\right)\mathcal{L}_E\left(s-\frac{1}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s,\chi^2\right)}{\mathcal{L}_F\left(s-\frac{1}{2},\chi\right)\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_E\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s+1,\chi^2\right)}$	1	1	1	1	1	1

Table B.2: Poles of $J(w, \chi_s)$ for $w \in W(P_E, H_E)$, E is a field

In the following table we list the Gindikin-Karpelevich factor $J(w, \chi, \lambda)$. Here $\lambda(t) = |t_1|_E^{s_1}|t_2|_F^{s_2}$.

$w \in W(P_E, H_E)$	$J(w, \chi, \lambda)$
$[]$	1
$[2]$	$\frac{\mathcal{L}_F(s_2,\chi)}{\mathcal{L}_F(s_2+1,\chi)}$
$[2,1]$	$\frac{\mathcal{L}_F(s_2,\chi)\mathcal{L}_E(s_1+s_2,\chi\circ\mathrm{Nm})}{\mathcal{L}_F(s_2+1,\chi)\mathcal{L}_E(s_1+s_2+1,\chi\circ\mathrm{Nm})}$
$[2,1,2]$	$\frac{\mathcal{L}_F(s_2,\chi)\mathcal{L}_E(s_1+s_2,\chi\circ\mathrm{Nm})\mathcal{L}_F\left(3s_1+2s_2,\chi^2\right)}{\mathcal{L}_F(s_2+1,\chi)\mathcal{L}_E(s_1+s_2+1,\chi\circ\mathrm{Nm})\mathcal{L}_F(3s_1+2s_2+1,\chi^2)}$

$w \in W(P_E, H_E)$	$J(w,\chi,\lambda)$
$[2,1,2,1]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_E(s_1+s_2,\chi\circ\mathrm{Nm})\,\mathcal{L}_F\left(3s_1+2s_2,\chi^2\right)\mathcal{L}_E(2s_1+s_2,\chi\circ\mathrm{Nm})}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_E(s_1+s_2+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(3s_1+2s_2+1,\chi^2)\,\mathcal{L}_E(2s_1+s_2+1,\chi\circ\mathrm{Nm})}$
$[2,1,2,1,2]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_E(s_1+s_2,\chi\circ\mathrm{Nm})\,\mathcal{L}_F\left(3s_1+2s_2,\chi^2\right)\mathcal{L}_E(2s_1+s_2,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(3s_1+s_2,\chi)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_E(s_1+s_2+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(3s_1+2s_2+1,\chi^2)\,\mathcal{L}_E(2s_1+s_2+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(3s_1+s_2+1,\chi)}$

Table B.3: $J(w,\chi,\lambda)$ for $w \in W(P_E, H_E)$, E is a field

In the following table we list the exponents $\mathfrak{Re}\left(w^{-1}\cdot\chi_s\right)$ for all $w \in W(P_E, H_E)$ at the points $s = \frac{1}{2}$ and $\frac{3}{2}$.

$w \in W(P_E, H_E)$	$s = \frac{1}{2}$	$s = \frac{3}{2}$
$[]$	$[0,1]$	$\mathrm{Nm}_{E/F}[1,0] + 3[0,1]$
$[2]$	$-[0,1]$	$\mathrm{Nm}_{E/F}[1,0]$
$[2,1]$	$-\mathrm{Nm}_{E/F}[1,0] - [0,1]$	$-\mathrm{Nm}_{E/F}[1,0]$
$[2,1,2]$	$-\mathrm{Nm}_{E/F}[1,0] - 2[0,1]$	$-\mathrm{Nm}_{E/F}[1,0] - 3[0,1]$
$[2,1,2,1]$	$-\mathrm{Nm}_{E/F}[1,0] - 2[0,1]$	$-2\,\mathrm{Nm}_{E/F}[1,0] - 3[0,1]$
$[2,1,2,1,2]$	$-\mathrm{Nm}_{E/F}[1,0] - [0,1]$	$-2\,\mathrm{Nm}_{E/F}[1,0] - 3[0,1]$

Table B.4: The exponents $\mathfrak{Re}\left(w^{-1}\cdot\chi_s\right)$ for $w \in W(P_E, H_E)$, E is a field

B.0.2 Quadratic Extension Case

Assume $E = F \times K$, where K is a field. For this case we denote

$$t = h_{\alpha_1}(t_1)\, h_{\alpha_2}(t_2)\, h_{\alpha_3\alpha_4^\sigma}(t_3) = h_{\alpha_1}(t_1)\, h_{\alpha_2}(t_2)\, h_{\alpha_3}(t_3)\, h_{\alpha_3}(t_3^\sigma)\,,$$

where $t_1, t_2 \in F^\times, t_3 \in K^\times$.

In the following table we list $w^{-1}\cdot\chi_s(t)$ for the various $w \in W(P_E, H_E)$.

$w \in W(P_E, H_E)$	$w^{-1}\cdot\chi_s(t)$
$[]$	$\chi(t_2)\frac{\lvert t_2\rvert_F^{s+\frac{3}{2}}}{\lvert t_1\rvert_F \lvert t_3\rvert_K}$
$[2]$	$\chi\left(\frac{t_1\,\mathrm{Nm}_{K/F}(t_3)}{t_2}\right)\frac{\lvert t_1\rvert_F^{s+\frac{1}{2}}\lvert t_3\rvert_K^{s+\frac{1}{2}}}{\lvert t_2\rvert_F^{s+\frac{3}{2}}}$
$[2,1]$	$\chi\left(\frac{\mathrm{Nm}_{K/F}(t_3)}{t_1}\right)\frac{\lvert t_3\rvert_K^{s+\frac{1}{2}}}{\lvert t_1\rvert_F^{s+\frac{1}{2}}\lvert t_2\rvert_F}$
$[2,3]$	$\chi\left(\frac{t_1 t_2}{\mathrm{Nm}_{K/F}(t_3)}\right)\frac{\lvert t_1\rvert_F^{s+\frac{1}{2}}\lvert t_2\rvert_F^{s-\frac{1}{2}}}{\lvert t_3\rvert_K^{s+\frac{1}{2}}}$
$[2,1,3]$	$\chi\left(\frac{t_2^2}{t_1\,\mathrm{Nm}_{K/F}(t_3)}\right)\frac{\lvert t_2\rvert_F^{2s}}{\lvert t_1\rvert_F^{s+\frac{1}{2}}\lvert t_3\rvert_K^{s+\frac{1}{2}}}$
$[2,3,2]$	$\chi\left(\frac{t_1^2}{t_2}\right)\frac{\lvert t_1\rvert_F^{2s}}{\lvert t_2\rvert_F^{s-\frac{1}{2}}\lvert t_3\rvert_K}$
$[2,1,3,2]$	$\chi\left(\frac{t_1\,\mathrm{Nm}_{K/F}(t_3)}{t_2^2}\right)\frac{\lvert t_1\rvert_F^{s-\frac{1}{2}}\lvert t_3\rvert_K^{s-\frac{1}{2}}}{\lvert t_2\rvert_F^{2s}}$
$[2,3,2,1]$	$\chi\left(\frac{t_2}{t_1^2}\right)\frac{\lvert t_2\rvert_F^{s+\frac{1}{2}}}{\lvert t_1\rvert_F^{2s}\lvert t_3\rvert_K}$
$[2,1,3,2,1]$	$\chi\left(\frac{\mathrm{Nm}_{K/F}(t_3)}{t_1 t_2}\right)\frac{\lvert t_3\rvert_K^{s-\frac{1}{2}}}{\lvert t_1\rvert_F^{s-\frac{1}{2}}\lvert t_2\rvert_F^{s+\frac{1}{2}}}$

$w \in W(P_E, H_E)$	$w^{-1} \cdot \chi_s(t)$
$[2,1,3,2,3]$	$\chi\left(\frac{t_1}{\mathrm{Nm}_{K/F}(t_3)}\right) \frac{\lvert t_1\rvert_F^{s-\frac{1}{2}}}{\lvert t_2\rvert_F \lvert t_3\rvert_K^{s-\frac{1}{2}}}$
$[2,1,3,2,1,3]$	$\chi\left(\frac{t_2}{t_1\,\mathrm{Nm}_{K/F}(t_3)}\right) \frac{\lvert t_2\rvert_F^{s-\frac{3}{2}}}{\lvert t_1\rvert_F^{s-\frac{1}{2}} \lvert t_3\rvert_K^{s-\frac{1}{2}}}$
$[2,1,3,2,1,3,2]$	$\chi\left(\frac{1}{t_2}\right) \frac{1}{\lvert t_1\rvert_F \lvert t_2\rvert_F^{s-\frac{3}{2}} \lvert t_3\rvert_K}$

Table B.5: $w^{-1} \cdot \chi_s$ for $w \in W(P_E, H_E)$,$E = F \times K$

In the following table we list the Gindikin-Karpelevich factor $J(w, \chi_s)$ and the poles of the global Gindikin-Karpelevich factor $J(w, \chi_s)$ for $\mathfrak{Re}(s) > 0$.

		$s = \frac{1}{2}$			$s = \frac{3}{2}$		$s = \frac{5}{2}$
$w \in W(P_E, H_E)$	$J(w, \chi_s)$	$\mathbb{1}$	χ_K	$\chi^2 = \mathbb{1}$	$\mathbb{1}$	χ_K	$\mathbb{1}$
$[]$	1	0	0	0	0	0	0
$[2]$	$\frac{\mathcal{L}_F\left(s+\frac{3}{2},\chi\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)}$	0	0	0	0	0	0
$[2,1]$	$\frac{\mathcal{L}_F\left(s+\frac{1}{2},\chi\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)}$	1	0	0	0	0	0
$[2,3]$	$\frac{\mathcal{L}_F\left(s+\frac{3}{2},\chi\right)\mathcal{L}_K\left(s+\frac{1}{2},\chi\circ\mathrm{Nm}\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_K\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)}$	1	1	0	0	0	0
$[2,1,3]$	$\frac{\mathcal{L}_F\left(s+\frac{1}{2},\chi\right)\mathcal{L}_K\left(s+\frac{1}{2},\chi\circ\mathrm{Nm}\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_K\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)}$	2	1	0	0	0	0

		$s=\frac{1}{2}$			$s=\frac{3}{2}$		$s=\frac{5}{2}$
$w \in W(P_E,H_E)$	$J(w,\chi_s)$	$\mathbb{1}$	χ_K	$\chi^2=\mathbb{1}$	$\mathbb{1}$	χ_K	$\mathbb{1}$
$[2,3,2]$	$\frac{\mathcal{L}_F(s+\frac{3}{2},\chi)\,\mathcal{L}_K(s+\frac{1}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s-\frac{1}{2},\chi)}{\mathcal{L}_F(s+\frac{5}{2},\chi)\,\mathcal{L}_K(s+\frac{3}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s+\frac{1}{2},\chi)}$	1	1	0	1	0	0
$[2,1,3,2]$	$\frac{\mathcal{L}_F(s+\frac{1}{2},\chi)\,\mathcal{L}_K(s+\frac{1}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s,\chi^2)}{\mathcal{L}_F(s+\frac{5}{2},\chi)\,\mathcal{L}_K(s+\frac{3}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s+1,\chi^2)}$	3	2	1	0	0	0
$[2,3,2,1]$	$\frac{\mathcal{L}_F(s+\frac{3}{2},\chi)\,\mathcal{L}_K(s+\frac{1}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s-\frac{1}{2},\chi)\,\mathcal{L}_F(2s,\chi^2)}{\mathcal{L}_F(s+\frac{5}{2},\chi)\,\mathcal{L}_K(s+\frac{3}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s+\frac{1}{2},\chi)\,\mathcal{L}_F(2s+1,\chi^2)}$	2	2	1	1	0	0
$[2,1,3,2,1]$	$\frac{\mathcal{L}_F(s-\frac{1}{2},\chi)\,\mathcal{L}_K(s+\frac{1}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s,\chi^2)}{\mathcal{L}_F(s+\frac{5}{2},\chi)\,\mathcal{L}_K(s+\frac{3}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s+1,\chi^2)}$	3	2	1	1	0	0
$[2,1,3,2,3]$	$\frac{\mathcal{L}_F(s+\frac{1}{2},\chi)\,\mathcal{L}_K(s-\frac{1}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s,\chi^2)}{\mathcal{L}_F(s+\frac{5}{2},\chi)\,\mathcal{L}_K(s+\frac{3}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s+1,\chi^2)}$	3	2	1	1	1	0
$[2,1,3,2,1,3]$	$\frac{\mathcal{L}_F(s-\frac{1}{2},\chi)\,\mathcal{L}_K(s-\frac{1}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s,\chi^2)}{\mathcal{L}_F(s+\frac{5}{2},\chi)\,\mathcal{L}_K(s+\frac{3}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s+1,\chi^2)}$	3	2	1	2	1	0
$[2,1,3,2,1,3,2]$	$\frac{\mathcal{L}_F(s-\frac{3}{2},\chi)\,\mathcal{L}_K(s-\frac{1}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s,\chi^2)}{\mathcal{L}_F(s+\frac{5}{2},\chi)\,\mathcal{L}_K(s+\frac{3}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s+1,\chi^2)}$	2	2	1	2	1	1

Table B.6: Poles of $J(w,\chi_s)$ for $w \in W(P_E,H_E)$,
$E = F \times K$

In the following table we list the Gindikin-Karpelevich factor $J(w,\chi,\lambda)$. Here $\lambda(t) = |t_1|_F^{s_1}\,|t_2|_E^{s_2}\,|t_3|_K^{s_3}$.

$w \in W(P_E,H_E)$	$J(w,\chi,\lambda)$
[]	1

$w \in W(P_E, H_E)$	$J(w, \chi, \lambda)$
$[2]$	$\frac{\mathcal{L}_F(s_2,\chi)}{\mathcal{L}_F(s_2+1,\chi)}$
$[2,1]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_F(s_1+s_2,\chi)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_F(s_1+s_2+1,\chi)}$
$[2,3]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})}$
$[2,1,3]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2,\chi)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+1,\chi)}$
$[2,3,2]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_2+2s_3,\chi)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_2+2s_3+1,\chi)}$
$[2,1,3,2]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2,\chi)\,\mathcal{L}_F\left(s_1+2s_2+2s_3,\chi^2\right)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+1,\chi)\,\mathcal{L}_F(s_1+2s_2+2s_3+1,\chi^2)}$
$[2,3,2,1]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_2+2s_3,\chi)\,\mathcal{L}_F\left(s_1+2s_2+2s_3,\chi^2\right)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_2+2s_3+1,\chi)\,\mathcal{L}_F(s_1+2s_2+2s_3+1,\chi^2)}$
$[2,1,3,2,1]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2,\chi)\,\mathcal{L}_F\left(s_1+2s_2+2s_3,\chi^2\right)\,\mathcal{L}_F(s_2+2s_3,\chi)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+1,\chi)\,\mathcal{L}_F(s_1+2s_2+2s_3+1,\chi^2)\,\mathcal{L}_F(s_2+2s_3,\chi)}$
$[2,1,3,2,3]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2,\chi)\,\mathcal{L}_F\left(s_1+2s_2+2s_3,\chi^2\right)\,\mathcal{L}_K(s_1+s_2+s_3,\chi\circ\mathrm{Nm})}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+1,\chi)\,\mathcal{L}_F(s_1+2s_2+2s_3+1,\chi^2)\,\mathcal{L}_K(s_1+s_2+s_3+1,\chi\circ\mathrm{Nm})}$
$[2,1,3,2,1,3]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2,\chi)\,\mathcal{L}_F\left(s_1+2s_2+2s_3,\chi^2\right)\,\mathcal{L}_F(s_2+2s_3,\chi)\,\mathcal{L}_K(s_1+s_2+s_3,\chi\circ\mathrm{Nm})}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+1,\chi)\,\mathcal{L}_F(s_1+2s_2+2s_3+1,\chi^2)\,\mathcal{L}_F(s_2+2s_3,\chi)\,\mathcal{L}_K(s_1+s_2+s_3+1,\chi\circ\mathrm{Nm})}$
$[2,1,3,2,1,3,2]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2,\chi)\,\mathcal{L}_F\left(s_1+2s_2+2s_3,\chi^2\right)\,\mathcal{L}_F(s_2+2s_3,\chi)\,\mathcal{L}_K(s_1+s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+2s_3,\chi)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+1,\chi)\,\mathcal{L}_F(s_1+2s_2+2s_3+1,\chi^2)\,\mathcal{L}_F(s_2+2s_3,\chi)\,\mathcal{L}_K(s_1+s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+2s_3,\chi)}$

Table B.7: $J(w, \chi, \lambda)$ for $w \in W(P_E, H_E)$, $E = F \times K$

In the following table we list the exponents $\mathfrak{Re}\,(w^{-1} \cdot \chi_s)$ for all $w \in W(P_E, H_E)$.

$w \in W(P_E, H_E)$	$s = \frac{1}{2}$	$s = \frac{3}{2}$
[]	$[0,1,0]$	$[1,0,0] + 3\,[0,1,0] + \mathrm{Nm}_{E/F}\,[0,0,1]$
[2]	$-\,[0,1,0]$	$[1,0,0] + \mathrm{Nm}_{E/F}\,[0,0,1]$
[2, 1]	$-\,[1,0,0] - [0,1,0]$	$-\,[1,0,0] + \mathrm{Nm}_{E/F}\,[0,0,1]$
[2, 3]	$-\,[0,1,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$	$[1,0,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$
[2, 1, 3]	$-\,[1,0,0] - [0,1,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$	$-\,[1,0,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$
[2, 3, 2]	$-\,[0,1,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$	$[1,0,0] - [0,1,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$
[2, 1, 3, 2]	$-\,[1,0,0] - 2\,[0,1,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$	$-\,[1,0,0] - 3\,[0,1,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$
[2, 3, 2, 1]	$-\,[1,0,0] - 2\,[0,1,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$	$-2\,[1,0,0] - [0,1,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$
[2, 1, 3, 2, 1]	$-\,[1,0,0] - 2\,[0,1,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$	$-2\,[1,0,0] - 3\,[0,1,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$
[2, 1, 3, 2, 3]	$-\,[1,0,0] - 2\,[0,1,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$	$-\,[1,0,0] - 3\,[0,1,0] - 2\,\mathrm{Nm}_{E/F}\,[0,0,1]$
[2, 1, 3, 2, 1, 3]	$-\,[1,0,0] - 2\,[0,1,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$	$-2\,[1,0,0] - 3\,[0,1,0] - 2\,\mathrm{Nm}_{E/F}\,[0,0,1]$
[2, 1, 3, 2, 1, 3, 2]	$-\,[1,0,0] - [0,1,0] - \mathrm{Nm}_{E/F}\,[0,0,1]$	$-2\,[1,0,0] - 3\,[0,1,0] - 2\,\mathrm{Nm}_{E/F}\,[0,0,1]$

Table B.8: The exponents $\mathfrak{Re}\,(w^{-1} \cdot \chi_s)$ for $w \in W(P_E, H_E)$, $E = F \times K$

B.0.3 Split Case

Assume $E = F \times F \times F$. For this case, we denote

$$t = h_{\alpha_1}(t_1)\, h_{\alpha_2}(t_2)\, h_{\alpha_3}(t_3)\, h_{\alpha_4}(t_4)\,,$$

where $t_1, t_2, t_3, t_4 \in F^{\times}$.

In the following table we list $w^{-1} \cdot \chi_s(t)$ for the various $w \in W(P_E, H_E)$ and also the resulting characters $w^{-1} \cdot \chi_s$.

$w \in W(P_E, H_E)$	$w^{-1} \cdot \chi_s(t)$
[]	$\chi(t_2) \frac{\lvert t_2\rvert^{s+\frac{3}{2}}}{\lvert t_1t_3t_4\rvert}$
[2]	$\chi\left(\frac{t_1t_3t_4}{t_2}\right) \frac{\lvert t_1t_3t_4\rvert^{s+\frac{1}{2}}}{\lvert t_2\rvert^{s+\frac{3}{2}}}$
[2, 1]	$\chi\left(\frac{t_3t_4}{t_1}\right) \frac{\lvert t_3t_4\rvert^{s+\frac{1}{2}}}{\lvert t_2\rvert\lvert t_1\rvert^{s+\frac{1}{2}}}$
[2, 3]	$\chi\left(\frac{t_1t_4}{t_3}\right) \frac{\lvert t_1t_4\rvert^{s+\frac{1}{2}}}{\lvert t_2\rvert\lvert t_3\rvert^{s+\frac{1}{2}}}$
[2, 4]	$\chi\left(\frac{t_1t_3}{t_4}\right) \frac{\lvert t_1t_3\rvert^{s+\frac{1}{2}}}{\lvert t_2\rvert\lvert t_4\rvert^{s+\frac{1}{2}}}$
[2, 1, 3]	$\chi\left(\frac{t_2t_4}{t_1t_3}\right) \frac{\lvert t_2\rvert^{s-\frac{1}{2}}\lvert t_4\rvert^{s+\frac{1}{2}}}{\lvert t_1t_3\rvert^{s+\frac{1}{2}}}$
[2, 1, 4]	$\chi\left(\frac{t_2t_3}{t_1t_4}\right) \frac{\lvert t_2\rvert^{s-\frac{1}{2}}\lvert t_3\rvert^{s+\frac{1}{2}}}{\lvert t_1t_4\rvert^{s+\frac{1}{2}}}$
[2, 3, 4]	$\chi\left(\frac{t_2t_1}{t_3t_4}\right) \frac{\lvert t_2\rvert^{s-\frac{1}{2}}\lvert t_1\rvert^{s+\frac{1}{2}}}{\lvert t_3t_4\rvert^{s+\frac{1}{2}}}$
[2, 1, 3, 2]	$\chi\left(\frac{t_4^2}{t_2}\right) \frac{\lvert t_4\rvert^{2s}}{\lvert t_1t_3\rvert\lvert t_2\rvert^{s-\frac{1}{2}}}$

$w \in W(P_E, H_E)$	$w^{-1} \cdot \chi_s(t)$
$[2,1,4,2]$	$\chi\left(\frac{t_3^2}{t_2}\right) \frac{\lvert t_3\rvert^{2s}}{\lvert t_1 t_4\rvert \lvert t_2\rvert^{s-\frac{1}{2}}}$
$[2,3,4,2]$	$\chi\left(\frac{t_1^2}{t_2}\right) \frac{\lvert t_1\rvert^{2s}}{\lvert t_3 t_4\rvert \lvert t_2\rvert^{s-\frac{1}{2}}}$
$[2,1,3,4]$	$\chi\left(\frac{t_2^2}{t_1 t_3 t_4}\right) \frac{\lvert t_2\rvert^{2s}}{\lvert t_1 t_3 t_4\rvert^{s+\frac{1}{2}}}$
$[2,3,4,2,1]$	$\chi\left(\frac{t_2}{t_1^2}\right) \frac{\lvert t_2\rvert^{s+\frac{1}{2}}}{\lvert t_3 t_4\rvert \lvert t_1\rvert^{2s}}$
$[2,1,4,2,3]$	$\chi\left(\frac{t_2}{t_3^2}\right) \frac{\lvert t_2\rvert^{s+\frac{1}{2}}}{\lvert t_1 t_4\rvert \lvert t_3\rvert^{2s}}$
$[2,1,3,2,4]$	$\chi\left(\frac{t_2}{t_4^2}\right) \frac{\lvert t_2\rvert^{s+\frac{1}{2}}}{\lvert t_1 t_3\rvert \lvert t_4\rvert^{2s}}$
$[2,1,3,4,2]$	$\chi\left(\frac{t_1 t_3 t_4}{t_2^2}\right) \frac{\lvert t_1 t_3 t_4\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert^{2s}}$
$[2,1,3,4,2,1]$	$\chi\left(\frac{t_3 t_4}{t_1 t_2}\right) \frac{\lvert t_3 t_4\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert^{s+\frac{1}{2}} \lvert t_1\rvert^{s-\frac{1}{2}}}$
$[2,1,3,4,2,3]$	$\chi\left(\frac{t_1 t_4}{t_2 t_3}\right) \frac{\lvert t_1 t_4\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert^{s+\frac{1}{2}} \lvert t_3\rvert^{s-\frac{1}{2}}}$
$[2,1,3,4,2,4]$	$\chi\left(\frac{t_1 t_3}{t_2 t_4}\right) \frac{\lvert t_1 t_3\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert^{s+\frac{1}{2}} \lvert t_4\rvert^{s-\frac{1}{2}}}$
$[2,1,3,4,2,1,3]$	$\chi\left(\frac{t_4}{t_1 t_3}\right) \frac{\lvert t_4\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert \lvert t_1 t_3\rvert^{s-\frac{1}{2}}}$
$[2,1,3,4,2,1,4]$	$\chi\left(\frac{t_3}{t_1 t_4}\right) \frac{\lvert t_3\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert \lvert t_1 t_4\rvert^{s-\frac{1}{2}}}$
$[2,1,3,4,2,3,4]$	$\chi\left(\frac{t_1}{t_3 t_4}\right) \frac{\lvert t_1\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert \lvert t_3 t_4\rvert^{s-\frac{1}{2}}}$

$w \in W(P_E, H_E)$	$w^{-1} \cdot \chi_s(t)$
$[2,1,3,4,2,1,3,4]$	$\chi\left(\frac{t_2}{t_1 t_3 t_4}\right) \frac{\lvert t_2\rvert^{s-\frac{3}{2}}}{\lvert t_1 t_3 t_4\rvert^{s-\frac{1}{2}}}$
$[2,1,3,4,2,1,3,4,2]$	$\chi\left(\frac{1}{t_2}\right) \frac{1}{\lvert t_2\rvert^{s-\frac{3}{2}} \lvert t_1 t_3 t_4\rvert}$

Table B.9: $w^{-1} \cdot \chi_s$ for $w \in W(P_E, H_E)$, $E = F \times F \times F$

In the following table we list the Gindikin-Karpelevich factor $J(w, \chi_s)$ and the poles of the global Gindikin-Karpelevich factor $J(w, \chi_s)$ for $\mathfrak{Re}(s) > 0$.

		$s = \frac{1}{2}$		$s = \frac{3}{2}$	$s = \frac{5}{2}$
$w \in W(P_E, H_E)$	$J(w, \chi_s)$	$\chi = \mathbb{1}$	$\chi^2 = \mathbb{1}$	$\chi = \mathbb{1}$	1
$[]$	1	0	0	0	0
$[2]$	$\frac{\mathcal{L}\left(s+\frac{3}{2},\chi\right)}{\mathcal{L}\left(s+\frac{5}{2},\chi\right)}$	0	0	0	0
$[2,1]$ $[2,3]$ $[2,4]$	$\frac{\mathcal{L}\left(s+\frac{1}{2},\chi\right)}{\mathcal{L}\left(s+\frac{5}{2},\chi\right)}$	1	0	0	0
$[2,1,3]$					

		$s=\frac{1}{2}$		$s=\frac{3}{2}$	$s=\frac{5}{2}$
$w \in W(P_E, H_E)$	$J(w, \chi_s)$	$\chi = \mathbb{1}$	$\chi^2 = \mathbb{1}$	$\chi = \mathbb{1}$	1
$[2,1,4]$ $[2,3,4]$	$\frac{\mathcal{L}\left(s+\frac{1}{2},\chi\right)^2}{\mathcal{L}\left(s+\frac{3}{2},\chi\right)\mathcal{L}\left(s+\frac{5}{2},\chi\right)}$	2	0	0	0
$[2,1,3,2]$ $[2,1,4,2]$ $[2,3,4,2]$	$\frac{\mathcal{L}\left(s+\frac{1}{2},\chi\right)\mathcal{L}\left(s-\frac{1}{2},\chi\right)}{\mathcal{L}\left(s+\frac{5}{2},\chi\right)\mathcal{L}\left(s+\frac{3}{2},\chi\right)}$	2	0	1	0
$[2,1,3,4]$	$\frac{\mathcal{L}\left(s+\frac{1}{2},\chi\right)^3}{\mathcal{L}\left(s+\frac{3}{2},\chi\right)^2\mathcal{L}\left(s+\frac{5}{2},\chi\right)}$	3	0	0	0
$[2,3,4,2,1]$ $[2,1,4,2,3]$ $[2,1,3,2,4]$	$\frac{\mathcal{L}\left(s-\frac{1}{2},\chi\right)\mathcal{L}\left(s+\frac{1}{2},\chi\right)\mathcal{L}\left(2s,\chi^2\right)}{\mathcal{L}\left(s+\frac{3}{2},\chi\right)\mathcal{L}\left(s+\frac{5}{2},\chi\right)\mathcal{L}(2s+1,\chi^2)}$	3	1	1	0
$[2,1,3,4,2]$	$\frac{\mathcal{L}\left(s+\frac{1}{2},\chi\right)^3\mathcal{L}\left(2s,\chi^2\right)}{\mathcal{L}\left(s+\frac{5}{2},\chi\right)\mathcal{L}\left(s+\frac{3}{2},\chi\right)^2\mathcal{L}(2s+1,\chi^2)}$	4	1	0	0
$[2,1,3,4,2,1]$ $[2,1,3,4,2,3]$ $[2,1,3,4,2,4]$	$\frac{\mathcal{L}\left(s-\frac{1}{2},\chi\right)\mathcal{L}\left(s+\frac{1}{2},\chi\right)^2\mathcal{L}\left(2s,\chi^2\right)}{\mathcal{L}\left(s+\frac{3}{2},\chi\right)^2\mathcal{L}\left(s+\frac{5}{2},\chi\right)\mathcal{L}(2s+1,\chi^2)}$	4	1	1	0

		$s=\frac{1}{2}$		$s=\frac{3}{2}$	$s=\frac{5}{2}$
$w \in W(P_E, H_E)$	$J(w,\chi_s)$	$\chi = \mathbb{1}$	$\chi^2 = \mathbb{1}$	$\chi = \mathbb{1}$	1
$[2,1,3,4,2,1,3]$ $[2,1,3,4,2,1,4]$ $[2,1,3,4,2,3,4]$	$\frac{\mathcal{L}\left(s-\frac{1}{2},\chi\right)^2 \mathcal{L}\left(s+\frac{1}{2},\chi\right) \mathcal{L}\left(2s,\chi^2\right)}{\mathcal{L}\left(s+\frac{3}{2},\chi\right)^2 \mathcal{L}\left(s+\frac{5}{2},\chi\right) \mathcal{L}(2s+1,\chi^2)}$	4	1	2	0
$[2,1,3,4,2,1,3,4]$	$\frac{\mathcal{L}\left(s-\frac{1}{2},\chi\right)^3 \mathcal{L}\left(2s,\chi^2\right)}{\mathcal{L}\left(s+\frac{3}{2},\chi\right)^2 \mathcal{L}\left(s+\frac{5}{2},\chi\right) \mathcal{L}(2s+1,\chi^2)}$	4	1	3	0
$[2,1,3,4,2,1,3,4,2]$	$\frac{\mathcal{L}\left(s-\frac{1}{2},\chi\right)^2 \mathcal{L}\left(s-\frac{3}{2},\chi\right) \mathcal{L}\left(2s,\chi^2\right)}{\mathcal{L}\left(s+\frac{5}{2},\chi\right) \mathcal{L}\left(s+\frac{3}{2},\chi\right)^2 \mathcal{L}(2s+1,\chi^2)}$	3	1	3	1

Table B.10: Poles of $J(w,\chi_s)$ for $w \in W(P_E, H_E)$, $E = F \times F \times F$

In the following table we list the Gindikin-Karpelevich factor $J(w,\chi,\lambda)$. Here $\lambda(t) = |t_1|_F^{s_1} |t_2|_F^{s_2} |t_3|_F^{s_3} |t_4|_F^{s_4}$.

$w \in W(P_E, H_E)$	$J(w,\chi,\lambda)$
$[]$	1
$[2]$	$\frac{\mathcal{L}(s_2,\chi)}{\mathcal{L}(s_2+1,\chi)}$
$[2,1]$	$\frac{\mathcal{L}(s_2,\chi) \mathcal{L}(s_1+s_2,\chi)}{\mathcal{L}(s_2+1,\chi) \mathcal{L}(s_1+s_2+1,\chi)}$

$w \in W(P_E, H_E)$	$J(w, \chi, \lambda)$
$[2,3]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)}$
$[2,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)}$
$[2,1,3]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)}$
$[2,1,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)}$
$[2,3,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)}$
$[2,1,3,2]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_1+s_2+s_3,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_1+s_2+s_3+1,\chi)}$
$[2,1,4,2]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}(s_1+s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)}$
$[2,3,4,2]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}(s_2+s_3+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)}$
$[2,1,3,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)}$
$[2,3,4,2,1]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}(s_2+s_3+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)}$
$[2,1,4,2,3]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}(s_1+s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)}$
$[2,1,3,2,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_1+s_2+s_3,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_1+s_2+s_3+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)}$
$[2,1,3,4,2]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)}$
$[2,1,3,4,2,1]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\mathcal{L}(s_2+s_3+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)}$

$w \in W(P_E, H_E)$	$J(w,\chi,\lambda)$
$[2,1,3,4,2,3]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\mathcal{L}(s_1+s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)}$
$[2,1,3,4,2,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\mathcal{L}(s_1+s_2+s_3,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_1+s_2+s_3+1,\chi)}$
$[2,1,3,4,2,1,3]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\mathcal{L}(s_2+s_3+s_4,\chi)\,\mathcal{L}(s_1+s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)}$
$[2,1,3,4,2,1,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\mathcal{L}(s_1+s_2+s_3,\chi)\,\mathcal{L}(s_2+s_3+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_1+s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)}$
$[2,1,3,4,2,3,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\mathcal{L}(s_1+s_2+s_3,\chi)\,\mathcal{L}(s_1+s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)}$
$[2,1,3,4,2,1,3,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\mathcal{L}(s_1+s_2+s_3,\chi)\,\mathcal{L}(s_2+s_3+s_4,\chi)\,\mathcal{L}(s_1+s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_1+s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)}$
$[2,1,3,4,2,1,3,4,2]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\mathcal{L}(s_1+s_2+s_3,\chi)\,\mathcal{L}(s_2+s_3+s_4,\chi)\,\mathcal{L}(s_1+s_2+s_4,\chi)\,\mathcal{L}(s_1+s_2+s_3+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_1+s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_3+s_4+1,\chi)}$

Table B.11: $J(w,\chi,\lambda)$ for $w \in W(P_E, H_E)$,
$E = F \times F \times F$

In the following table we list the exponents $\mathfrak{Re}\left(w^{-1}\cdot\chi_s\right)$ for all $w \in W(P_E, H_E)$.

$w \in W(P_E, H_E)$	$s = \frac{1}{2}$	$s = \frac{3}{2}$
$[]$	$[0,1,0,0]$	$[1,3,1,1]$
$[2]$	$-[0,1,0,0]$	$[1,0,1,1]$

$w \in W\,(P_E, H_E)$	$s = \frac{1}{2}$	$s = \frac{3}{2}$
$[2,1]$	$-[1,1,0,0]$	$[-1,0,1,1]$
$[2,3]$	$-[0,1,1,0]$	$[1,0,-1,1]$
$[2,4]$	$-[0,1,0,1]$	$[1,0,1,-1]$
$[2,1,3]$	$-[1,1,1,0]$	$[-1,0,-1,1]$
$[2,1,4]$	$-[1,1,0,1]$	$[-1,0,1,-1]$
$[2,3,4]$	$-[0,1,1,1]$	$[1,0,-1,-1]$
$[2,1,3,2]$	$-[1,1,1,0]$	$[-1,-1,-1,1]$
$[2,1,4,2]$	$-[1,1,0,1]$	$[-1,-1,1,-1]$
$[2,3,4,2]$	$-[0,1,1,1]$	$[1,-1,-1,-1]$
$[2,1,3,4]$	$-[1,1,1,1]$	$[-1,0,-1,-1]$
$[2,3,4,2,1]$	$-[1,1,1,1]$	$[-2,-1,-1,-1]$
$[2,1,4,2,3]$	$-[1,1,1,1]$	$[-1,-1,-2,-1]$
$[2,1,3,2,4]$	$-[1,1,1,1]$	$[-1,-1,-1,-2]$
$[2,1,3,4,2]$	$-[1,2,1,1]$	$[-1,-3,-1,-1]$

$w \in W(P_E, H_E)$	$s = \frac{1}{2}$	$s = \frac{3}{2}$
$[2,1,3,4,2,1]$	$-[1,2,1,1]$	$[-2,-3,-1,-1]$
$[2,1,3,4,2,3]$	$-[1,2,1,1]$	$[-1,-3,-2,-1]$
$[2,1,3,4,2,4]$	$-[1,2,1,1]$	$[-1,-3,-1,-2]$
$[2,1,3,4,2,1,3]$	$-[1,2,1,1]$	$[-2,-3,-2,-1]$
$[2,1,3,4,2,1,4]$	$-[1,2,1,1]$	$[-2,-3,-1,-2]$
$[2,1,3,4,2,3,4]$	$-[1,2,1,1]$	$[-1,-3,-2,-2]$
$[2,1,3,4,2,1,3,4]$	$-[1,2,1,1]$	$[-2,-3,-2,-2]$
$[2,1,3,4,2,1,3,4,2]$	$-[1,1,1,1]$	$[-2,-3,-2,-2]$

Table B.12: The exponents $\mathfrak{Re}(w^{-1} \cdot \chi_s)$ for $w \in W(P_E, H_E)$, $E = F \times F \times F$

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